

City University of New York (CUNY)

CUNY Academic Works

Open Educational Resources

Queensborough Community College

2018

College Algebra Through Problem Solving (2018 Edition)

Danielle Cifone

CUNY Queensborough Community College

Karan Puri

CUNY Queensborough Community College

Debra Masklanko

CUNY Queensborough Community College

Ewa Dabkowska

CUNY Queensborough Community College

[How does access to this work benefit you? Let us know!](#)

More information about this work at: https://academicworks.cuny.edu/qb_oers/23

Discover additional works at: <https://academicworks.cuny.edu>

This work is made publicly available by the City University of New York (CUNY).

Contact: AcademicWorks@cuny.edu

College Algebra

through

Problem Solving

2018 Edition

Danielle Cifone

Karan Puri

Debra Maslanko

Ewa Dabkowska

This book is partially an outgrowth of the second author's work with Jonathan Cornick and G. Michael Guy. We thank them for providing valuable insight along the way. We also thank Robert Holt for helpful suggestions and edits that he made to working drafts of the book. This work was funded by the Office of Academic Affairs at Queensborough Community College. We thank them for their support.

Danielle Cifone dcifone@qcc.cuny.edu

Karan Puri karan@karanpuri.net <http://www.karanpuri.net>

Debra Maslanko dmaslanko@qcc.cuny.edu

Ewa Dabkowska edabkowska@qcc.cuny.edu

This work is licensed under the Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License. To view a copy of this license, visit

<http://creativecommons.org/licenses/by-nc-sa/4.0/>.



Table of Contents

Section 1: Linear and Absolute Value Equations	5
<i>Linear Equations.....</i>	<i>5</i>
<i>The Distributive Property.....</i>	<i>6</i>
<i>Introduction to Absolute Value</i>	<i>9</i>
<i>Equations involving the Absolute Value</i>	<i>10</i>
Section 2: Linear and Compound Inequalities	16
<i>Graphing Inequalities and Interval Notation.....</i>	<i>16</i>
<i>Solving Linear Inequalities.....</i>	<i>18</i>
<i>Compound Inequalities.....</i>	<i>19</i>
Compound Inequalities Involving AND (intersection).....	19
Compound Inequalities Involving OR (union)	23
Section 3: Functions.....	26
<i>Function Notation.....</i>	<i>26</i>
<i>Graphs of Functions.....</i>	<i>28</i>
<i>Domain and Range of a Function's graph</i>	<i>30</i>
<i>Intercepts</i>	<i>31</i>
<i>Obtaining information from graphs.</i>	<i>32</i>
Section 4: Linear Functions and Slope	35
<i>Definition of Slope</i>	<i>35</i>
<i>Graphing Using the Slope and the y-intercept.....</i>	<i>38</i>
<i>Equations of Horizontal and Vertical Lines.....</i>	<i>41</i>
Section 5: Finding the Equations of Lines	43
<i>Parallel and Perpendicular Lines</i>	<i>46</i>
<i>Writing the Equation of a Line that is Perpendicular to a Given Line.....</i>	<i>48</i>
Section 6: Solving Systems of Equations Graphically	50
Section 7: Solving Systems of Equations Algebraically	58
Section 8: Integral Exponents.....	65
<i>The Product Rule</i>	<i>65</i>
<i>The Quotient Rule.....</i>	<i>66</i>
<i>The Zero Power Rule.....</i>	<i>67</i>

<i>Negative Exponents</i>	69
<i>The Power Raised to a Power Rule (or The Power Rule)</i>	71
<i>Product Raised to a Power Rule</i>	71
<i>Quotient Raised to a Power Rule</i>	72
Section 9: Greatest Common Factors and Factoring by Grouping	74
<i>Adding or Subtracting Polynomials</i>	74
<i>Multiplying Polynomials</i>	75
<i>Dividing Polynomials</i>	76
<i>Factoring Out the Greatest Common Factor (GCF)</i>	78
<i>Factoring Out a Greatest Binomial Factor</i>	79
<i>Factoring by Grouping</i>	81
Section 10: Factoring Trinomials	84
<i>Multi-Step Factoring</i>	88
Section 11: Factoring Special Forms	90
<i>Factoring the Difference of Two Squares</i>	90
<i>Multi-Step Factoring</i>	93
<i>Factoring the Sum or Difference of Two Cubes</i>	95
Section 12: Solving Polynomial Equations by Factoring	98
<i>Zero Product Property</i>	98
Section 13: Rational Functions (Simplifying, Multiplying and Dividing)	106
<i>Domain of a Rational Function</i>	106
<i>Simplifying Rational Expressions</i>	108
<i>Multiplying Rational Expressions</i>	109
<i>Dividing Rational Expressions</i>	109
<i>Adding and Subtracting Rational Expressions</i>	112
Section 14: Complex Rational Expressions	116
<i>Simplifying Complex Rational Expressions</i>	116
Section 15: Rational Equations	119
<i>Solving Rational Equations</i>	119
Section 16: Literal Equations	125

Section 17: Radical Expressions and Functions.....	129
<i>Finding the Domain of a Square Root Function.....</i>	<i>131</i>
<i>Evaluating Radical Functions.....</i>	<i>132</i>
<i>Simplifying Radical Expressions.....</i>	<i>133</i>
Section 18: Rational Exponents.....	135
<i>Simplifying Expressions with Rational Exponents.....</i>	<i>138</i>
Section 19: Simplifying Radical Expressions	141
<i>Multiplying Radical Expressions</i>	<i>143</i>
<i>Adding and Subtracting Radical Expressions.....</i>	<i>144</i>
<i>Multiplying Multiple Radical Expressions.....</i>	<i>146</i>
Section 20: Rationalizing Denominators.....	149
<i>Rationalizing Denominators Containing One Term</i>	<i>149</i>
<i>Rationalizing Denominators that Contain Two Terms</i>	<i>151</i>
Section 21: Radical Equations	154
<i>Cube roots and equations with two radical expressions</i>	<i>157</i>
Section 22: Complex Numbers	160
Section 23: Completing the Square	166
<i>The Square Root Property</i>	<i>166</i>
<i>Perfect Square Trinomials</i>	<i>166</i>
Section 24: The Quadratic Formula	171
<i>Pythagorean Theorem.....</i>	<i>175</i>
Section 25: Graphing Quadratic Functions	178
Section 26: Exponential and Logarithmic Functions	190
<i>Graphs of Exponential Functions.....</i>	<i>191</i>
<i>Logarithmic Functions</i>	<i>196</i>
<i>Inverse Properties of Logarithms.....</i>	<i>198</i>
<i>Graphs of Logarithmic Functions.....</i>	<i>199</i>
Section 27: Properties of Logarithms	200
<i>Change of Base Formula & using the calculator to find logarithms.....</i>	<i>206</i>

Section 28: Exponential & Logarithmic Equations	208
<i>Exponential Equations.....</i>	<i>208</i>
<i>Logarithmic Equations.....</i>	<i>213</i>

Section 1: Linear and Absolute Value Equations

Linear Equations

We begin our exploration by discussing linear equations in one variable. An example of such an equation, and the question that we may be asked is below:

Problem 1.1

Solve for x . Check your solution.

$$4x + 1 = 2x + 7$$

Since our goal here is to solve for x , we must move all the variable (x) terms to one side, and all the terms without variables (constant terms) to the other side. We can begin to do this in any of four different ways. However, once we have chosen our first step, the subsequent steps are fixed for us. For now, we will move the $2x$ to the left side with the $4x$. We will revisit the other three options later.

$$\begin{aligned} 4x + 1 &= 2x + 7 \\ 4x - \mathbf{2x} + 1 &= 2x - \mathbf{2x} + 7 \\ 2x + 1 &= 7 \\ 2x + 1 - \mathbf{1} &= 7 - \mathbf{1} \\ 2x &= 6 \\ \frac{2x}{2} &= \frac{6}{2} \\ x &= 3 \end{aligned}$$

This is the original equation.

Subtract $2x$ from both sides.

Simplify.

Subtract 1 from both sides.

Simplify.

Divide both sides by 2.

Simplify.

Check:

To check, we will substitute our solution into the *original equation*. If we are correct, then we will get a true statement.

$$\begin{aligned} 4x + 1 &= 2x + 7 \\ 4(3) + 1 &\stackrel{?}{=} 2(3) + 7 \\ 12 + 1 &\stackrel{?}{=} 6 + 7 \\ 13 &= 13 \quad \checkmark \end{aligned}$$

Since we have arrived at a true statement $13 = 13$, our answer is correct.

Therefore, $x = 3$.

■

In the previous problem, we noted that there were three other ways to begin. Next, we see each of these options. Notice that we reach the same solution in all cases.

First, subtract 1 from both sides.

$$\begin{aligned} 4x + 1 &= 2x + 7 \\ 4x + 1 - 1 &= 2x + 7 - 1 \\ 4x &= 2x + 6 \\ 4x - 2x &= 2x - 2x + 6 \\ 2x &= 6 \\ \frac{2x}{2} &= \frac{6}{2} \\ x &= 3 \end{aligned}$$

First, subtract $4x$ from both sides.

$$\begin{aligned} 4x + 1 &= 2x + 7 \\ 4x - 4x + 1 &= 2x - 4x + 7 \\ 1 &= -2x + 7 \\ 1 - 7 &= -2x + 7 - 7 \\ -6 &= -2x \\ \frac{-6}{-2} &= \frac{-2x}{-2} \\ 3 &= x \\ x &= 3 \end{aligned}$$

First, subtract 7 from both sides.

$$\begin{aligned} 4x + 1 &= 2x + 7 \\ 4x + 1 - 7 &= 2x + 7 - 7 \\ 4x - 6 &= 2x \\ 4x - 4x - 6 &= 2x - 4x \\ -6 &= -2x \\ \frac{-6}{-2} &= \frac{-2x}{-2} \\ 3 &= x \\ x &= 3 \end{aligned}$$

We may use any of the above to arrive at the correct answer.

The Distributive Property

Some equations involve multiplication with parentheses and we must understand the distributive property to solve these problems. For example,

$$3(4x + 3) = 12x + 9$$

To remove these parentheses, we use the distributive property defined below:

The Distributive Property

$$a(b + c) = ab + ac$$

In other words, when we multiply a quantity outside the parentheses by two (or more) quantities that are being added (or subtracted) inside the parentheses, we must multiply the quantity outside by each of the quantities inside, turn by turn.

Problem 1.2

Solve for x . Check your solution.

$$3(3x + 7) = -5x + 7$$

We begin by applying the distributive property on the left hand side. Then, we proceed as usual.

$3(3x + 7) = -5x + 7$	This is the original equation.
$3(3x) + 3(7) = -5x + 7$	Apply the distributive property.
$9x + 21 = -5x + 7$	Simplify.
$9x + 5x + 21 = -5x + 5x + 7$	Add $5x$ to both sides.
$14x + 21 = 7$	Simplify.
$14x + 21 - 21 = 7 - 21$	Subtract 21 from both sides.
$14x = -14$	Simplify.
$\frac{14x}{14} = \frac{-14}{14}$	Divide both sides by 14.
$x = -1$	Simplify.

Check:

To check, we will still substitute our solution into the *original equation*. If we are correct, then we will get a true statement.

$$\begin{aligned}
 3(3x + 7) &= -5x + 7 \\
 3(3(-1) + 7) &\neq -5(-1) + 7 \\
 3(-3 + 7) &\neq 5 + 7 \\
 3(4) &\neq 12 \\
 12 &= 12 \quad \checkmark
 \end{aligned}$$

In our check, notice how we first simplified inside the parentheses. This is due to the Order of Operations, which tells us to simplify inside parentheses first.

Problem 1.3Solve for t . Check your solution.

$$2t + 4 = t + 10$$

Problem 1.4Solve for q . Check your solution.

$$-2q - 7 = 3 + 2q$$

Problem 1.5Solve for x . Check your solution.

$$-2(2x + 6) = -3 + 5x$$

Problem 1.6Solve for x . Check your solution.

$$3 + 3(x + 8) = 4(2x + 3) + 5$$

Introduction to Absolute Value

The Absolute Value

The absolute value of a number x is defined to be the distance of x from 0. In symbols, we write the absolute value of x as $|x|$.

Using the above definition, we see that $|4| = 4$ because the distance of the number 4 from 0 is 4. Similarly, $|-4| = 4$ as well, because the distance of -4 from 0 is also 4.

To familiarize yourself with absolute values, answer the following:

Problem 1.7

Evaluate:

- (a) $|7| =$
- (b) $|-5| =$
- (c) $|-13.45| =$
- (d) $|-3458| =$
- (e) $|0| =$

Notice that whenever the number is positive (or zero), the absolute value is equal to the number and whenever the number is negative, we have to change the sign of the number to get the absolute value. Recall that changing the sign of a number is the same as taking its negative. This leads us to a different way of thinking about the absolute value.

The Absolute Value

An alternative way of defining the absolute value is the following:

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

By the above definition, if we wish to calculate $|17|$, we will use the first half of the definition, because 17 is positive. Therefore, we make no change and $|17| = 17$.

However, if we want to calculate $|-17|$, we must use the second half of the definition, since -17 is negative. So, we see that $|-17| = -(-17) = 17$.

Now, let us use the above definition to answer the following:

Problem 1.8

Evaluate:

- (a) $|8| =$
 (b) $|-7| =$
 (c) $|-5.309| =$
 (d) $|0| =$
 (e) $|-123| =$

Equations involving the Absolute Value

Now, we turn our attention to solving equations involving the absolute value. Consider the examples below:

Problem 1.9

Find the solution set of the equation.

$$|x| = 3$$

Solution

Here, we are being asked to find the set of all possible numbers, whose absolute value is 3. We know that there can only be two such numbers, namely 3 and -3 .

Therefore, we have two solutions, namely $x = 3$ and $x = -3$. Our solution set is $\{3, -3\}$.

■

Problem 1.10

Find the solution set of the equation. Check each solution.

$$|3x + 2| = 7$$

Solution

Notice that the expression inside the absolute value symbol is now $3x + 2$. Therefore, there are two possibilities. Either $3x + 2 = 7$, or $3x + 2 = -7$. These two equations must be solved separately now for us to get our solution set.

$$\begin{array}{rcl}
 3x + 2 & = & 7 \\
 3x + 2 - 2 & = & 7 - 2 \\
 3x & = & 5 \\
 \frac{3x}{3} & = & \frac{5}{3} \\
 x & = & \frac{5}{3}
 \end{array}$$

$$\begin{array}{rcl}
 3x + 2 & = & -7 \\
 3x + 2 - 2 & = & -7 - 2 \\
 3x & = & -9 \\
 \frac{3x}{3} & = & \frac{-9}{3} \\
 x & = & -3
 \end{array}$$

Therefore, our solution set is $\{\frac{5}{3}, -3\}$.

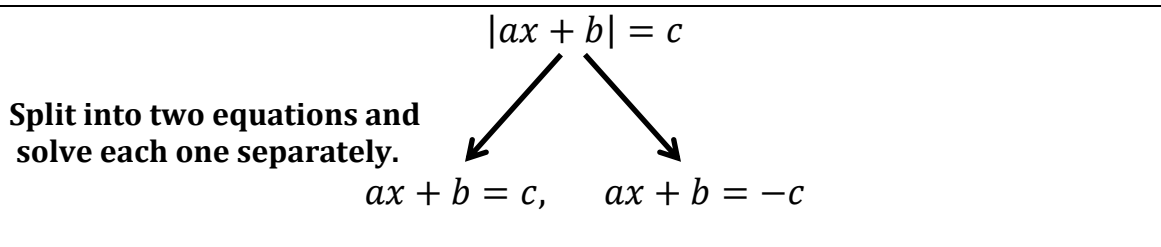
Check:

To check, we will still substitute *each of our solutions* into the *original equation*. If we are correct, then we will get a true statement.

$$\begin{aligned}
 |3x + 2| &= 7 \\
 \left| 3\left(\frac{5}{3}\right) + 2 \right| &\neq 7 \\
 |5 + 2| &\neq 7 \\
 |7| &\neq 7 \\
 7 &= 7 \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 |3x + 2| &= 7 \\
 |3(-3) + 2| &\neq 7 \\
 |-9 + 2| &\neq 7 \\
 |-7| &\neq 7 \\
 7 &= 7 \quad \checkmark
 \end{aligned}$$

The main idea above is that any equation involving absolute values will eventually split into two equations. We will then solve those two equations separately to find our solutions.

**Problem 1.11**

Find the solution set of the equation.
Check each solution.

$$|2x + 1| = 5$$

Problem 1.12

Find the solution set of the equation.
Check each solution.

$$|7x + 4| = 32$$

Now, consider the example below:

Problem 1.13

Find the solution set of the equation. Check each solution.

$$3|4x + 1| - 8 = 7$$

Here, not only do we have an absolute value, we also have other terms that are being multiplied by and subtracted from the absolute value term. Our goal here will be to **first isolate the absolute value**, and then **split into two equations and solve**.

Solution

$$3|4x + 1| - 8 = 7$$

$$3|4x + 1| - 8 + 8 = 7 + 8$$

$$3|4x + 1| = 15$$

$$\frac{3|4x + 1|}{3} = \frac{15}{3}$$

$$|4x + 1| = 5$$

Now, we have isolated the absolute value and we can proceed as before by splitting into two equations.

$$\begin{aligned} 4x + 1 &= 5 \\ 4x + 1 - 1 &= 5 - 1 \\ 4x &= 4 \\ \frac{4x}{4} &= \frac{4}{4} \\ x &= 1 \end{aligned}$$

$$\begin{aligned} 4x + 1 &= -5 \\ 4x + 1 - 1 &= -5 - 1 \\ 4x &= -6 \\ \frac{4x}{4} &= \frac{-6}{4} \\ x &= -\frac{3}{2} \end{aligned}$$

Therefore, our solution set is $\left\{1, -\frac{3}{2}\right\}$. Now, check each of these solutions yourself.

Check:

Problem 1.14

Find the solution set of the equation.
Check each solution.

$$|3x + 4| - 5 = 9$$

Problem 1.15

Find the solution set of the equation.
Check each solution.

$$5|4x - 3| = 15$$

Problem 1.16

Find the solution set of the equation.
Check each solution.

$$7|5x - 4| + 3 = 31$$

Problem 1.17

Find the solution set of the equation.
Check each solution.

$$|3x| - 4 = 8$$

Sometimes, we may face an equation with two absolute values. For example,

Problem 1.18

Find the solution set of the equation. Check each solution.

$$|3x - 8| = |5x - 6|$$

Solution

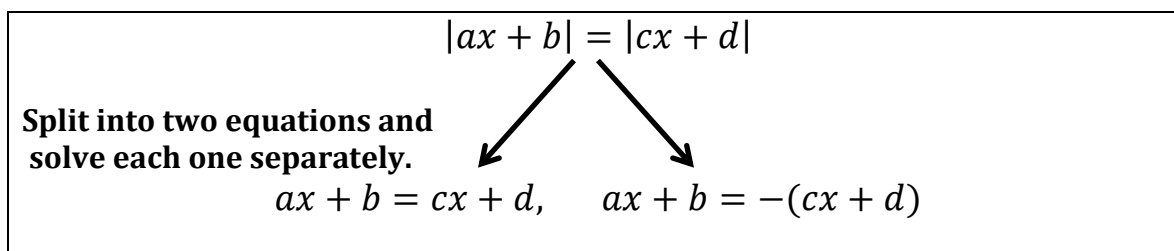
Notice that since the absolute values of two expressions are equal, this gives us two possibilities. Either the two expressions are equal or one is the negative of the other. Therefore, we split our equation into two as follows:

$$\begin{aligned} 3x - 8 &= 5x - 6 \\ 3x - \mathbf{5x} - 8 &= 5x - \mathbf{5x} - 6 \\ -2x - 8 &= -6 \\ -2x - 8 + \mathbf{8} &= -6 + \mathbf{8} \\ -2x &= 2 \\ \frac{-2x}{-2} &= \frac{2}{-2} \\ x &= -1 \end{aligned}$$

$$\begin{aligned} 3x - 8 &= -(5x - 6) \\ 3x - 8 &= -5x + 6 \\ 3x + \mathbf{5x} - 8 &= -5x + \mathbf{5x} + 6 \\ 8x - 8 &= 6 \\ 8x - 8 + \mathbf{8} &= 6 + \mathbf{8} \\ 8x &= 14 \\ \frac{8x}{8} &= \frac{14}{8} \\ x &= \frac{7}{4} \end{aligned}$$

Therefore, our solution set is $\{-1, \frac{7}{4}\}$. Now, *check each solution* by substituting each solution turn by turn into the *original equation*.

The main idea above is that any equation involving two absolute values will still split into two equations, just as before. We will then solve those two equations separately to find our solutions.



Problem 1.19

Find the solution set of the equation.
Check each solution.

$$|9x - 3| = |7x + 11|$$

Problem 1.20

Find the solution set of the equation.
Check each solution.

$$|7x - 14| = |10x + 1|$$

Section 2: Linear and Compound Inequalities

A **linear inequality** is simply a linear relation that involves $<$, \leq , $>$, or \geq . These symbols express one side of the inequality being larger or smaller than the other side.

$x > -3$ “ x is any value greater than -3 ” But $x \neq -3$	$x < 9$ “ x is any value less than 9 ” But $x \neq 9$
$x \geq 2$ “ x is any value greater than OR equal to 2 ”	$x \leq -5$ “ x is any value less than OR equal to -5 ”

Note: In each inequality, the variable is on the left hand side. This is no coincidence! When written this way, the solution becomes easier to comprehend. If the variable happens to be on the right hand side, we should read the inequality “backwards”

For Example:

$$4 < x$$

is equivalent to

$$x > 4$$

$$-7 \geq x$$

is equivalent to

$$x \leq -7$$

$$0 > x$$

is equivalent to

$$x < 0$$

Graphing Inequalities and Interval Notation

Often, solutions to linear inequalities are shown graphed on a number line and written in interval notation. Graphing an inequality on a number line first allows us to more easily translate our solution into **interval notation**.

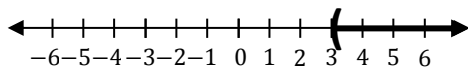
An inequality contains an infinite number of solutions and interval notation is a way of showing the solutions by **only listing the smallest and largest values**. This may be a concept you are used to, but let's review.

Problem 2.1

Graph the inequalities $x > 3$ and $x \leq -2$ on a number line and express in interval notation.

Solution

$$x > 3$$



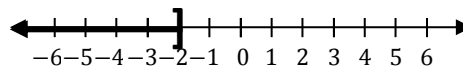
We use parentheses on the number line when the inequality involves $<$ or $>$ to show the value **IS NOT** part of the solution.

The number line is shaded to the right to include all values greater than 3. The shaded portion starts at 3 and continues to infinity.

In interval notation, we write:

$$(3, \infty)$$

$$x \leq -2$$



We use brackets on the number line when the inequality involves \leq or \geq to show the value **IS** part of the solution.

The number line is shaded to the left to include all values less than -2 . The shaded portion starts at negative infinity and continues to -2 .

In interval notation, we write:

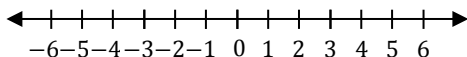
$$(-\infty, -2]$$

Note: We will always use parentheses around ∞ or $-\infty$ since these are values we will never reach.

Problem 2.2

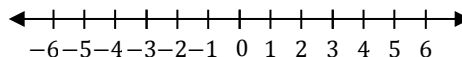
Graph the inequality and express in interval notation.

$$x \geq -5$$

**Problem 2.3**

Graph the inequality and express in interval notation.

$$x < 1$$



Solving Linear Inequalities

When solving linear inequalities, we take the same steps as when solving linear equations, being careful to include the inequality in each step.

Problem 2.4

Solve for x . Graph the solution on a number line and express in interval notation.

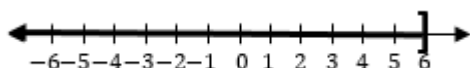
$$6x - 5 \leq 3x + 13$$

Solution

When solving an inequality, it is beneficial to keep the variable terms on the left hand side of the inequality, if possible. Doing so will make the final solution easier to understand. With that said, our first step will be to subtract $3x$ from both sides.

$$\begin{aligned} 6x - 5 &\leq 3x + 13 \\ 6x - 3x - 5 &\leq 3x - 3x + 13 \\ 3x - 5 &\leq 13 \\ 3x - 5 + 5 &\leq 13 + 5 \\ 3x &\leq 18 \\ \frac{3x}{3} &\leq \frac{18}{3} \\ x &\leq 6 \end{aligned}$$

" x is less than or equal to 6"



$$(-\infty, 6]$$

Original equation
Subtract $3x$ from both sides
Simplify
Add 5 to both sides
Simplify
Divide both sides by 3
Simplify

Solution graphed on a number line

Solution in interval notation

Dividing (or multiplying) by a negative number will reverse the inequality

Sometimes this cannot be avoided but let's look at why it happens.

$$-2 < 6$$

Divide each side by -2

$$\frac{-2}{-2} ? \frac{6}{-2}$$

Simplify

$$1 > -3$$

The inequality **MUST** be reversed to make the statement true.

Problem 2.5

Solve and graph the solution set on a number line. Express your answer in interval notation.

$$-2x + 9 < 3x - 11$$

Problem 2.6

Solve and graph the solution set on a number line. Express your answer in interval notation.

$$7(x + 2) \leq 4x - 1$$

Compound Inequalities

A **compound inequality** is formed by joining two inequalities with the word *AND* or the word *OR*. In this section we will take a closer look at the similarities and differences between these types of inequalities. As in the previous examples, we will express our solutions on a number line and in interval notation.

Compound Inequalities Involving *AND* (intersection)

A value is a solution to a compound inequality joined by the word *AND* if the value is a solution to BOTH inequalities.

The **intersection** of sets A and B, written $A \cap B$, is the set of all elements common to BOTH set A **AND** set B.

Note: If two sets do not have any elements in common, the intersection is expressed as a **null**, or empty, set.
 $\{ \}$ or \emptyset

Problem 2.7

Graph the solution set on a number line and express in interval notation.

$$-3x + 1 \geq 7 \text{ AND } 2x + 3 < 13$$

Solution:

When solving a compound inequality involving *AND*...

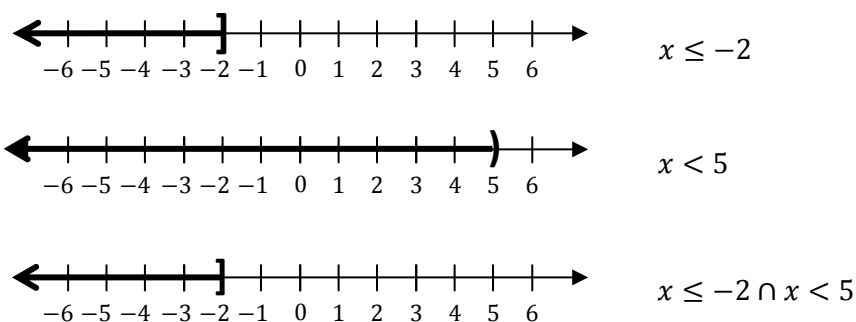
Solve each inequality separately

Graph the solutions to each inequality on a number line

Express the intersection in interval notation (the shaded portion that appears on **BOTH** number lines)

$-3x + 1 \geq 7$ $-3x \geq 6$ $x \leq -2$ <p>(remember to reverse inequality when dividing by a negative number)</p>		$2x + 3 < 13$ $2x < 10$ $x < 5$
--	--	---------------------------------

To find the intersection of these two inequalities, we graph each inequality separately and look for the section of the number line that is shaded on BOTH number lines. The third number line shows our solution.



The solution set is the set of all real numbers less than or equal to -2: $(-\infty, -2]$

Note: It is not always necessary to create three number lines. We may graph both inequalities on the same number line and express the solution as the interval where the number lines overlap.

Some compound inequalities involving AND can be written in a more shorter/condensed form. This form allows us to solve both inequalities at once by performing the same operations to all three parts of the inequality in order to isolate x in the middle.

Problem 2.8

Solve the inequality. Graph the solution set on a number line and express in interval notation.

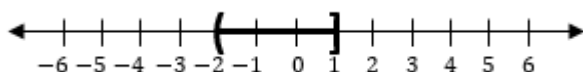
$$-5 < 2x - 1 \leq 1$$

Solution:

Since our goal is to isolate x in the middle of the two inequalities, we will begin by adding 1 to all three parts of the compound inequality. Ultimately, we want the value of x to be between two real numbers.

$-5 < 2x - 1 \leq 1$	This is the original equation
$-5 + 1 < 2x - 1 + 1 \leq 1 + 1$	Add 1 to all three parts
$-4 < 2x \leq 2$	Simplify
$\frac{-4}{2} < \frac{2x}{2} \leq \frac{2}{2}$	Divide all three parts by 2
$-2 < x \leq 1$	Simplify

The solution set contains all real numbers greater than -2 and less than or equal to 1. In other words, all real numbers between -2 (excluding -2) and 1 (including 1).



Solution in interval notation: $(-2, 1]$

Problem 2.9

Solve the inequality. Graph the solution set on a number line and express in interval notation.

$$2x \geq 5x - 15 \text{ AND } 7x > 2x + 10$$

Problem 2.10

Solve and graph the solution set on a number line.
Express your answer in interval notation.

$$-11 < 2x - 3 \leq -5$$

Problem 2.11

Solve and graph the solution set on a number line.
Express your answer in interval notation.

$$-4 < \frac{3x + 4}{2} \leq 3$$

Problem 2.12

Solve and graph the solution set on a number line. Express your answer in interval notation.

$$4 - 4x < -6 \text{ AND } \frac{x - 7}{5} \leq -2$$

Compound Inequalities Involving OR (union)

A value is a solution to a compound inequality joined by the word *OR* if the value is a solution to EITHER inequality.

The **union** of sets A and B, written $A \cup B$, is the set of all elements that are EITHER in set A **OR** set B (or both sets).

Note: The union of nonempty sets will never be empty.

Problem 2.13

Graph the solution set on a number line and express in interval notation.

$$5x - 2 < -17 \text{ OR } 1 \leq 4x + 1$$

Solution

When solving a compound inequality involving *OR*...

Solve each inequality separately

Graph the solutions to each inequality on a number line

Express the union in interval notation (**any** portion of the number line that is shaded)

$$5x - 2 < -17$$

$$5x < -15$$

$$x < -3$$

$$1 \leq 4x + 1$$

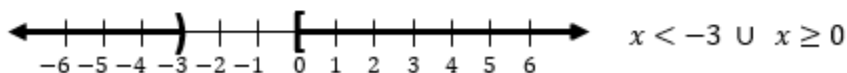
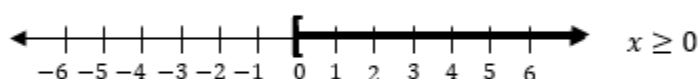
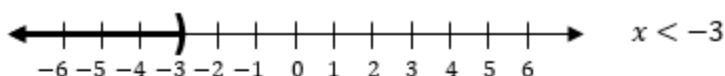
$$0 \leq 4x$$

$$0 \leq x$$

$$x \geq 0$$

(reverse inequality)

To find the union of these two inequalities, we graph each inequality separately. Our solution is any part of the number line that is shaded. The third number line shows our solution.



The solution set is the set of all real numbers less than -3 OR all real numbers greater than or equal to 0: $(-\infty, -3) \cup [0, \infty)$

Problem 2.14

Solve and graph the solution set on a number line. Express the solution set in interval notation.

$$5x + 3 < 18 \text{ OR } -2x - 7 \geq -5$$

Problem 2.15

Solve and graph the solution set on a number line. Express the solution set in interval notation.

$$-11 \leq 5x + 4 \text{ OR } 1 - 4x > 9$$

Problem 2.16

Solve and graph the solution set on a number line. Express the solution set in interval notation.

$$-12 \leq 2x + 1 < -7$$

Problem 2.17

Solve and graph the solution set on a number line. Express the solution set in interval notation.

$$2x + 1 > 4x - 3 \text{ AND } x - 1 \geq 3x + 5$$

Problem 2.18

Solve and graph the solution set on a number line. Express the solution set in interval notation.

$$\frac{5x + 1}{2} \geq 3 \text{ OR } 2x - 5 \leq -9$$

Problem 2.19

Solve and graph the solution set on a number line. Express the solution set in interval notation.

$$-10 < \frac{2x}{5} - 8 \leq -5$$

Section 3: Functions

What is a function?

A function is a relation between a set of inputs and a set of outputs, such that to each input value, there relates at most one output value. Functions are usually given in terms of algebraic expressions.

Determining whether a given equation represents a function:

$$y = 2x + 5$$

In order to determine if the given equation describes a function, we must check if for each value of x the equation assigns a unique value to y .

Check:

$$\text{when } x = 1, \text{ then } y = 2(1) + 5 = 7$$

$$\text{when } x = 5, \text{ then } y = 2(5) + 5 = 15$$

$$\text{when } x = -1, \text{ then } y = 2(-1) + 5 = 3$$

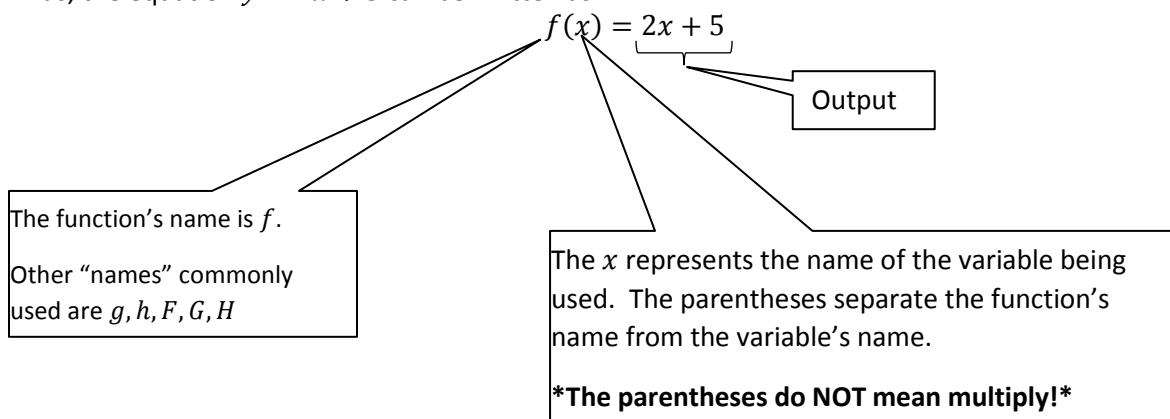
Since for any value of x , there is only one value of y , then this equation describes a function. In other words we could say, “ y is a function of x ”.

The y coordinate represents the output, or **dependent variable**, meaning the values of y depend upon what is substituted into the function for the input variable, or **independent variable**, in this case, x .

Function Notation

Equations that follow this definition can be written in **function notation**.

Thus, the equation $y = 2x + 5$ can be written as:



Problem 3.1

Evaluate $f(6)$ for the function $f(x)$.

$$f(x) = 2x^2 + x - 7$$

Solution

Our goal is to find the function's value when x is equal to 6. To do this, we need to replace every x with the value 6. When evaluating a function it is helpful to place parentheses around the value that is being substituted.

$$f(x) = 2x^2 + x - 7$$

This is the original function.

$$f(\quad) = 2(\quad)^2 + (\quad) - 7$$

Remove x 's and put parentheses instead

$$f(6) = 2(6)^2 + (6) - 7$$

Inside each parentheses write the value of

$$f(6) = 2(36) + 6 - 7$$

6

$$f(6) = 72 + 6 - 7$$

Evaluate using PEMDAS

$$f(6) = 78 - 7$$

$$f(6) = 71$$

The answer is $f(6) = 71$. We read this by saying " f of 6 is equal to 71."

Problem 3.2

Evaluate $g(4)$ for the function $g(x)$.

$$g(x) = -3x + 5$$

Problem 3.3

Evaluate $h(-2)$ for the function $h(x)$.

$$h(x) = x^2 - 2x + 1$$

Problem 3.4

Evaluate $F(2)$ for the function $F(x)$.

$$F(x) = -x^2 + 3x + 2$$

Problem 3.5

Evaluate $H(4)$ for the function $H(x)$.

$$H(x) = \frac{2x^2 - x + 5}{x - 3}$$

Problem 3.6

Evaluate $f(2a)$ for the function $f(x)$.

$$f(x) = 2x^2 + x - 1$$

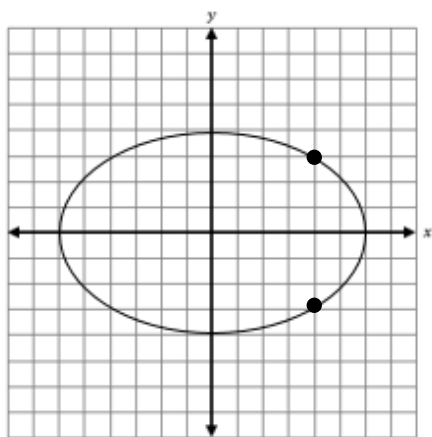
Problem 3.7

Evaluate $F(a + 1)$ for the function $F(x)$.

$$F(x) = 5x - 2$$

Graphs of Functions

In this section we will learn how to determine whether or not a graph represents a function. Let's start by considering the following graph:



One way to determine whether the given graph is a function is to see if for every x there is a unique y value. For example: when $x = 4$, then $y = 3$ and $y = -3$.

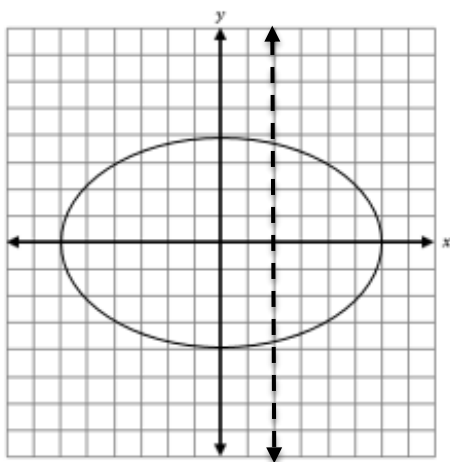
Thus, this graph is not a function because $x = 4$ corresponds to two different y values.

Another way to determine whether a graph represents a function is to use the **Vertical Line Test**.

The Vertical Line Test

If any vertical line intersects a graph in more than one point,
the graph does not define y as a function of x

Now, let's use the vertical line test to determine if the given graph represents a function.

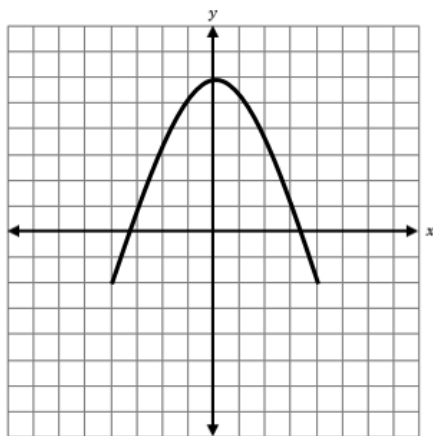


By graphing a vertical line (let's use $x = 2$), we see that it intersects the graph in two places, and thus, this graph is **NOT** a function.

Applying this test gives a quick and effective visual to decide if the graph is a function.

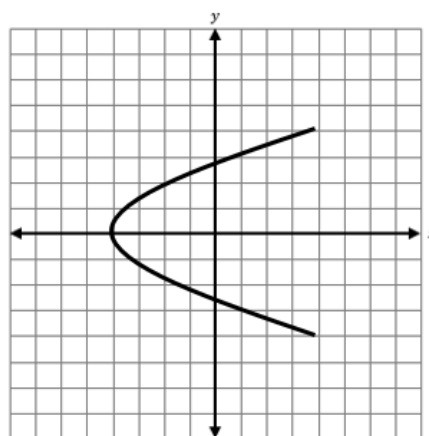
Problem 3.8

Determine whether the given graph is a function.



Problem 3.9

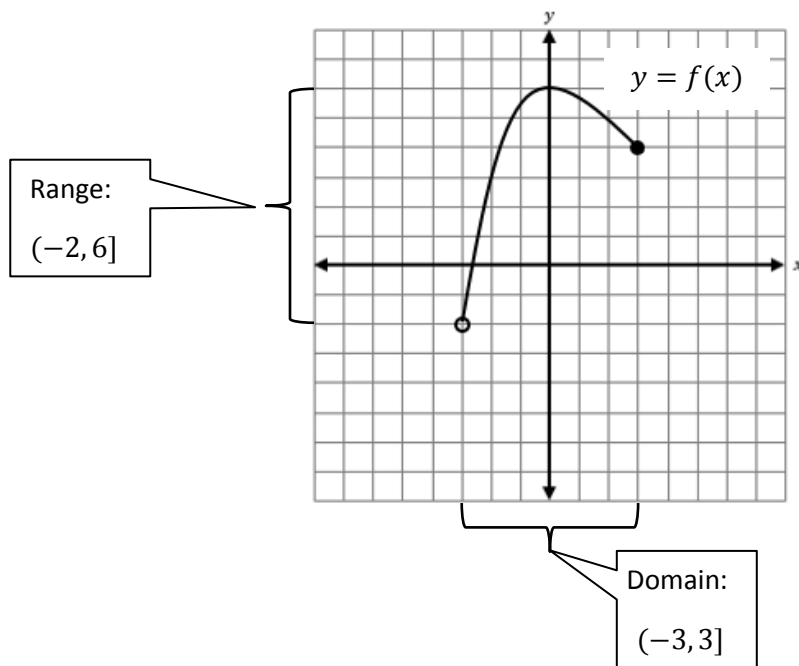
Determine whether the given graph is a function.



Domain and Range of a Function's graph

Domain is the set of all possible input values, that is, x -values.

Range is the set of all possible output values, that is, y -values.



(open dot) or “(,)” (parentheses) means that the point does not belong to the graph.

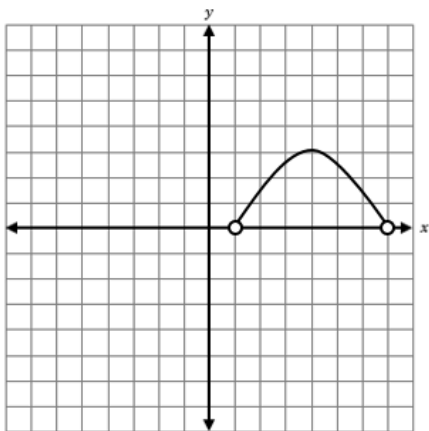
(closed dot) or “[,]” (brackets) means that the point belongs to the graph.

(arrow) or ∞ (infinity symbol) means that the graph extends to infinity.

Problem 3.10

Use the graph of the function to identify its domain and range in interval notation.

Solution



To find the **domain** always examine your graph from left to right. The first value of the domain is where the graph starts on the x -axis and the second value is where the graph ends on the x -axis. Using interval notation, **the function's domain is $(1, 7)$** .

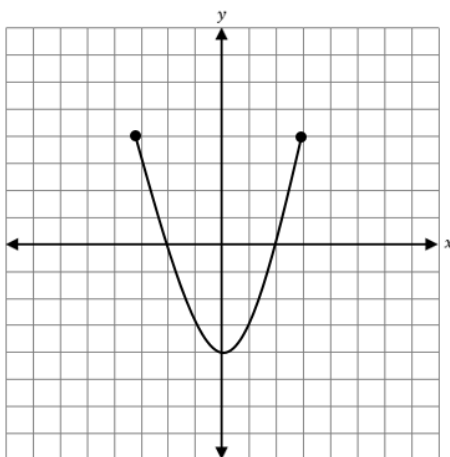
To find the **range** always examine your graph from bottom to top. The first value of the range is where the graph starts on the y -axis and the second value is where the graph ends on the y -axis. Using interval notation, **the function's range is $(0, 3]$** .

Intercepts

An **x -intercept** of a graph is the x -coordinate of a point where the graph intercepts the x -axis.

A **y -intercept** of a graph is the y -coordinate of a point where the graph intercepts the y -axis.

Identify the x - and y -intercepts for the given graph.



Solution

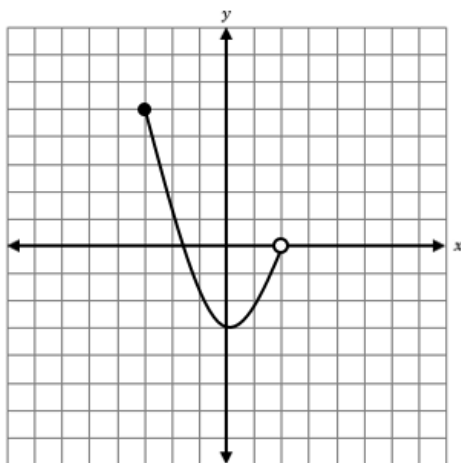
The graph crosses the x -axis at $(-2, 0)$ and at $(2, 0)$, so the **x -intercepts are -2 , and 2** .

The graph crosses the y -axis at $(0, -4)$, so the **y -intercept is -4** .

NOTE: A function can have multiple x -intercepts but a function cannot have more than one y -intercept.

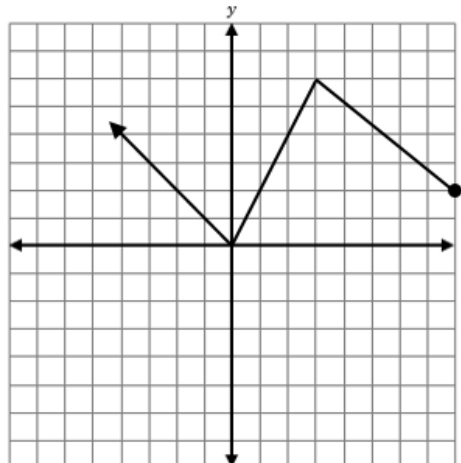
Problem 3.11

Use the graph of the function to identify its domain and range in interval notation and the coordinates of any x - and y -intercepts.



Problem 3.12

Use the graph of the function to identify its domain and range in interval notation and the coordinates of any x - and y -intercepts.



Obtaining information from graphs.

Problem 3.13

Use the graph to determine the following:

$$f(2) = ?$$

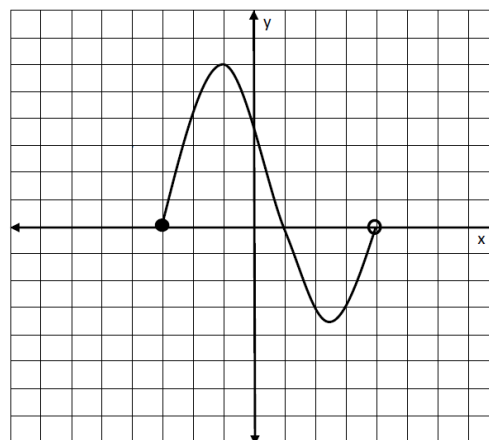
$$f(0) = ?$$

For what values of x is $f(x) = 6$?

For what values of x is $f(x) = -3$?

For what values of x is y positive?

Express in interval notation.



Solution

To find $f(2)$, we need to locate 2 on the x -axis. From this point, we look to the y -axis to find the corresponding y -value. We see that the y -value is equal to -3 .

Thus, $f(2) = -3$.

To find $f(0)$, we need to locate 0 on the x -axis. From this point, we look to the y -axis to find the corresponding y -value. We see that the y -value is equal to 4. Thus, $f(0) = 4$.

To find the value of x for which $f(x) = 6$, we need to locate 6 on the y -axis. From this point, we look to the x -axis to find the corresponding x -value. We see that the x -value is -1 .

Thus, $f(-1) = 6$.

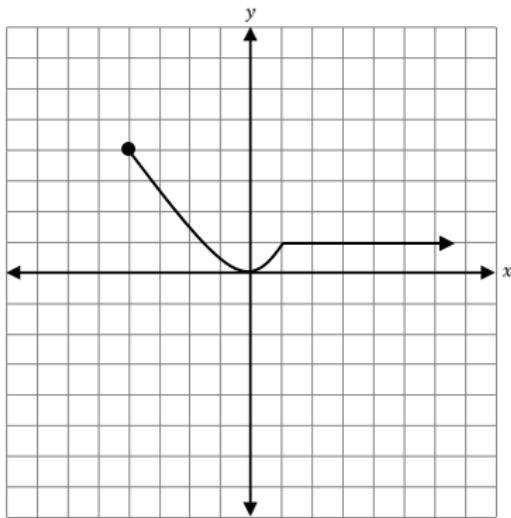
To find the value of x for which $f(x) = -3$, we need to locate -3 on the y -axis. From this point, we look to the x -axis to find the corresponding x -value. We see that there are **two** x -values, 2 and 3. Thus, $f(2) = -3$ and $f(3) = -3$.

The y values are positive above the x -axis. To find the x values that correspond to a positive y value, we look for the domain of only the part of the graph that lies above the x -axis. y is positive when x is any value between -3 and 1 . In interval notation, we express our solution as $(-3, 1)$.

NOTE: We use parentheses in the interval because when $x = -3$ or 1 , $y = 0$ and 0 is neither negative nor positive!

Problem 3.14

Use the graph of the function to determine the following:



$$f(-2) =$$

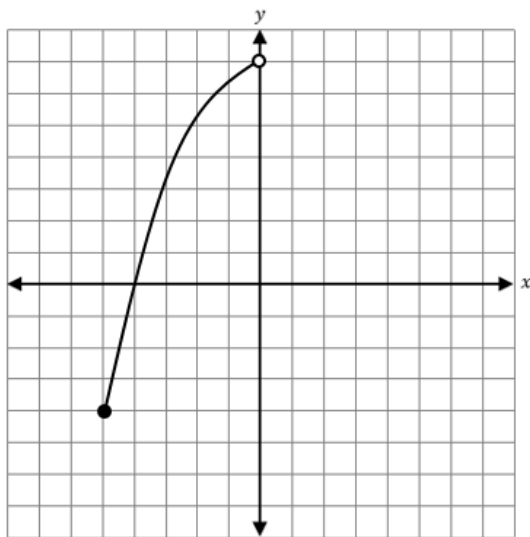
$$f(3) =$$

For what values of x is $f(x) = 0$?

For what values of x is $f(x) = 4$?

Problem 3.15

Use the graph of the function to determine the following:



Domain:

Range:

$$f(-5) =$$

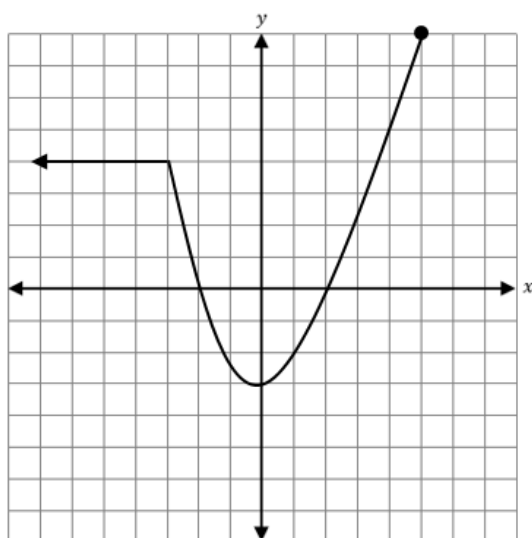
For what values of x is $f(x) = 6$?

Determine the coordinates of the x - and y -intercepts.

For what values of x is y positive?
Express in interval notation.

Problem 3.16

Use the graph of the function to determine the following:



Domain:

Range:

$$f(-4) =$$

$$f(1) =$$

For what values of x is $f(x) = 5$?

Determine the coordinates of the x - and y - intercepts.

For what values of x is y negative?
Express in interval notation.

Section 4: Linear Functions and Slope

There are three forms of the equation of a line:

The **standard form** of the equation of a line: $Ax + By = C$

Example: $2x + 3y = 6$

The **slope-intercept form** of the equation of a line: $y = mx + b$, where m represents the slope of the line and b represents the y-intercept of the line.

Example: $y = -\frac{2}{3}x + 2$

The **point-slope form** of the equation of a line: $y - y_1 = m(x - x_1)$

Example: $y - 2 = 5(x - 3)$

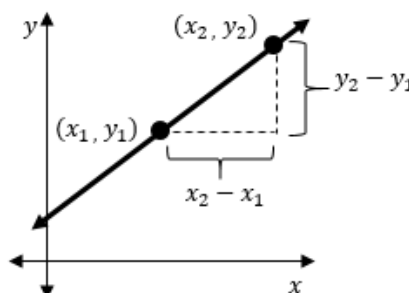
We've been talking about slope, but what does slope actually mean? Slope is a measure of the steepness of a line that compares the vertical change (or *rise*) to the horizontal change (or *run*) when we move from one point on the line to another point on the same line.

Definition of Slope

If we consider two distinct points on a line, $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ we can define the slope, m , of the line as follows:

$$m = \frac{\text{Change in } y}{\text{Change in } x} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

where $x_2 - x_1 \neq 0$.



To help remember the slope formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$, it is helpful to think of the following:

The y-axis is the **vertical** axis, and *rise* is a **vertical** motion.

The x-axis is the **horizontal** axis and *run* is a **horizontal** motion.

Using the expression $\frac{\text{rise}}{\text{run}}$ helps us to remember that the y's go into the numerator and the x's go into the denominator.

Sometimes the slope is an integer value and sometimes it is a fraction. The same applies to the y-intercept.

Problem 4.1

Let's try using the formula to calculate the slope of the line passing through the following points:

$$(-3, -4) \text{ and } (-1, 6)$$

Let $P_1 = (-3, -4)$ and $P_2 = (-1, 6)$
 So, $\overset{P_1}{(-3, -4)}$ and $\overset{P_2}{(-1, 6)}$
 $\overset{x_1, y_1}{x_1, y_1}$ $\overset{x_2, y_2}{x_2, y_2}$

First label the points.
 Then label the x and y coordinates accordingly.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - (-4)}{-1 - (-3)}$$

Substitute values into the slope formula.

$$m = \frac{6 - (-4)}{-1 - (-3)} = \frac{6 + 4}{-1 + 3} = \frac{10}{2} = 5$$

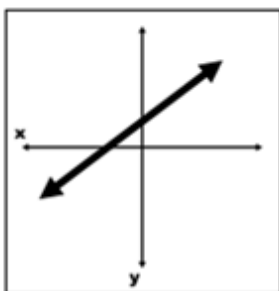
Simplify completely.

$$m = 5$$

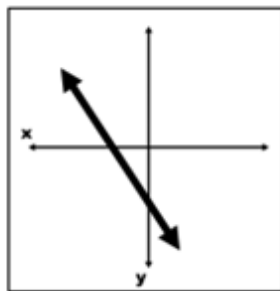
In the problem above, the slope ended up being a positive 5, so we say that the line has a *positive* slope. If the slope ends up being a negative number, then the line has a *negative* slope.

Sometimes after we do the calculation, the numerator ends up being zero, so the slope of the line is *zero*. Horizontal lines have zero slope.

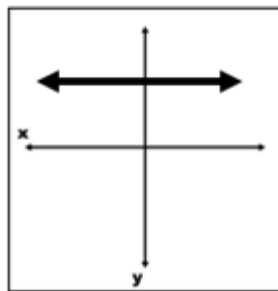
Other times, the denominator ends up being zero. Remember, we are not allowed to have zero in any denominator as we cannot divide by zero (*try this in your calculator!*) When this happens, the slope is said to be *undefined*. Vertical lines have undefined slope.



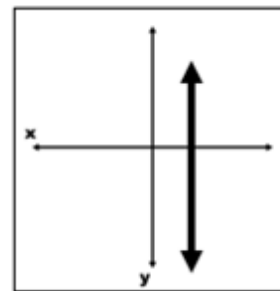
A line with a **positive slope**
 $(m > 0)$ will *rise*
 from left to right.



A line with a **negative slope**
 $(m < 0)$ will *fall*
 from left to right.



A line with a **slope of zero**
 $(m = 0)$ is a
horizontal line.



A line with a slope that is **undefined** is a
vertical line.

Problem 4.2

Find the slope of the line passing through the following points, *simplifying completely*:

$$(-2, 2) \text{ and } (8, -3)$$

Problem 4.3

Find the slope of the line passing through the following points, *simplifying completely*:

$$(-1, 3) \text{ and } (-4, 5)$$

Problem 4.4

Find the slope of the line passing through the following points, *simplifying completely*:

$$(3, 5) \text{ and } (-2, 5)$$

Problem 4.5

Find the slope of the line passing through the following points, *simplifying completely*:

$$(9, 3) \text{ and } (9, -5)$$

The **slope-intercept form** of a line is very useful since it tells us a lot about the line before we even try to graph it – we know what kind of slope it has, and where the line crosses the y -axis. Therefore, it is important to become adept at changing from standard form to slope-intercept form. We do this simply by taking the standard form of the equation and solving for y .

Let's practice changing the following from standard form to slope-intercept form and then identifying the slope and y -intercept of the line.

Problem 4.6

Solve for y . After solving, list the slope and the y -intercept.

$$5x - 2y = 8$$

Remember, our goal is to solve for y , so we must isolate y by itself on one side of the equation. We must do whatever is necessary to accomplish this:

$$\begin{aligned} 5x - 2y &= 8 \\ 5x - 5x - 2y &= -5x + 8 \\ -2y &= -5x + 8 \end{aligned}$$

This is the original equation.
Subtract $5x$ from both sides.
Simplify.

$$\frac{-2}{-2}y = \frac{-5}{-2}x + \frac{8}{-2}$$

Divide both sides by the coefficient of y ,
 -2 .

$$y = \frac{5}{2}x - 4$$

Simplify.

$$\begin{aligned} \text{Slope: } m &= \frac{5}{2} \\ y\text{-intercept: } &(0, -4) \end{aligned}$$

List the slope and the y -intercept of the
line.

Graphing Using the Slope and the y -intercept (or the $\frac{\text{Rise}}{\text{Run}}$ Method)

This is the best method of graphing a line when the equation of the line is in slope intercept, or $y = mx + b$ form. Follow these steps:

Identify the y -intercept (" b ") and plot the point in the y -axis. This is the point $(0, b)$.

Find the second point by using the slope, m . Write m as a fraction (integers values should be expressed as $\frac{m}{1}$), and use rise over run, starting at the y -intercept to plot the point.

Once the second point is found, use rise over run starting at that point in order to obtain a third point. (This third point is a "checkpoint", to make sure we have a straight line.)

Connect the three points to draw the line.

Problem 4.7

Graph the line whose equation is $-2x + y = -4$.

$$\begin{aligned} -2x + y &= -4. \\ -2x + 2x + y &= +2x - 4. \\ y &= 2x - 4. \end{aligned}$$

Solve for y .

Looking at the equation $y = 2x - 4$, we can see that -4 is the y -intercept. We plot the point $(0, -4)$ on the y -axis.

Identify the y -intercept (" b ") and plot the point in the y -axis.

Looking at the equation $y = 2x - 4$, we can see that 2 is the slope of the line. We should express the slope as a fraction:

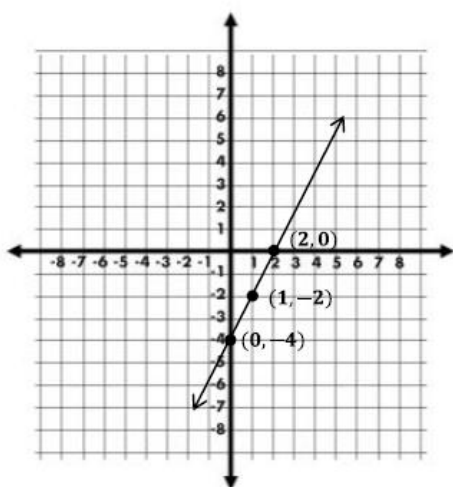
$$m = 2 = \frac{2}{1} = \frac{\text{rise}}{\text{run}}$$

Find the second point by using the slope, m . Write m as a fraction and use rise over run, starting at the y -intercept to plot the point.

Plot the point starting at the y -intercept, $(0, -4)$, moving 2 units *up* (the rise) and 1 unit to the *right* (the run). This will bring you to the point $(1, -2)$.

Now, starting at $(1, -2)$, move 2 units *up* (the rise) and 1 unit to the *right* (the run). This will bring you to the point $(2, 0)$.

Use rise over run starting at the second point in order to obtain a third point.

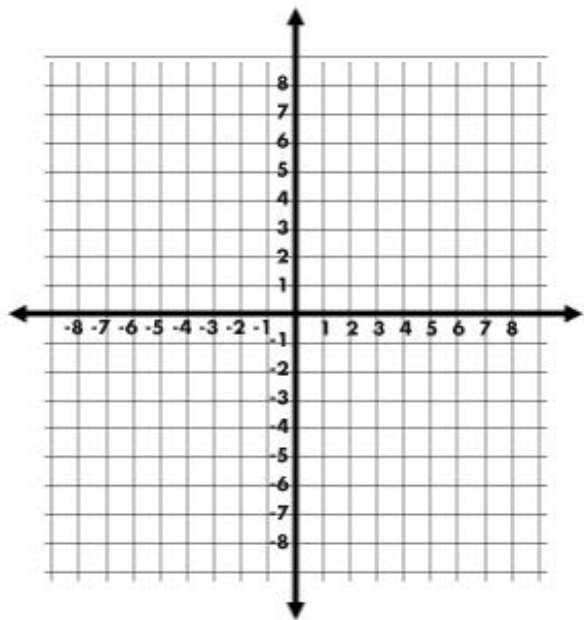


Connect the three points.

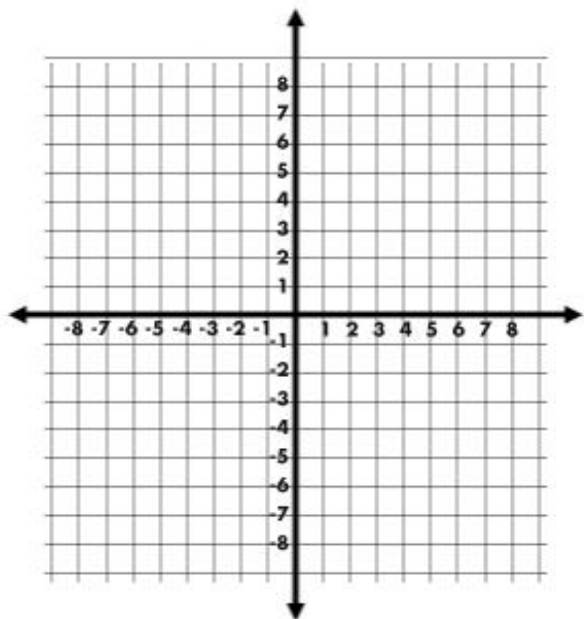
Important tip: If the slope is negative, put the negative sign into the numerator, and then think of the rise as *going down*. It may be easier to think about going up or down for the rise first, and then going right for the run.

Problem 4.8

Graph the line whose equation is $2x + 3y = 15$

**Problem 4.9**

Graph the line whose equation is $3x - y = 7$.



Equations of Horizontal and Vertical Lines

We mentioned previously that horizontal lines have zero slope: $m = 0$. If that is the case, then the equation $y = mx + b$ becomes $y = (0)x + b$, which simplifies to $y = b$, where b is the y -intercept. **All horizontal lines are of the form $y = b$.** Horizontal lines pass the vertical line test, so they are functions. They are often called *constant functions*.

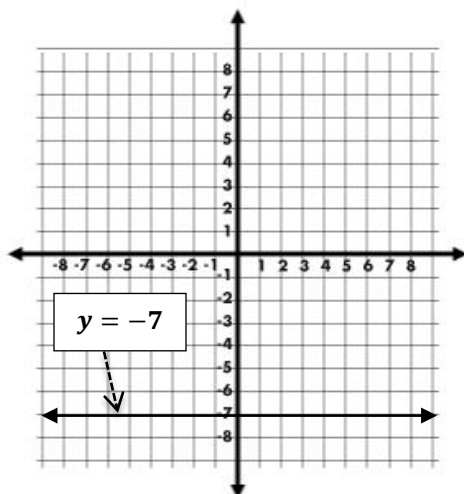
Graphing a Horizontal Line

Problem 4.10

Graph $y = -7$ on the $x - y$ coordinate plane.

Solution

Since -7 is b , the y -intercept, we can plot that point $(0, -7)$ on the y -axis. Now draw a horizontal line through that point in order to graph the line $y = -7$.



Vertical lines are of the form $x = a$, where a is the x -intercept.

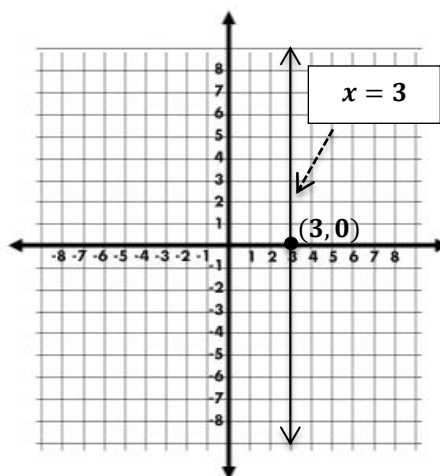
Graphing a Vertical Line

Problem 4.11

Graph $x = 3$ on the $x - y$ coordinate plane.

Solution

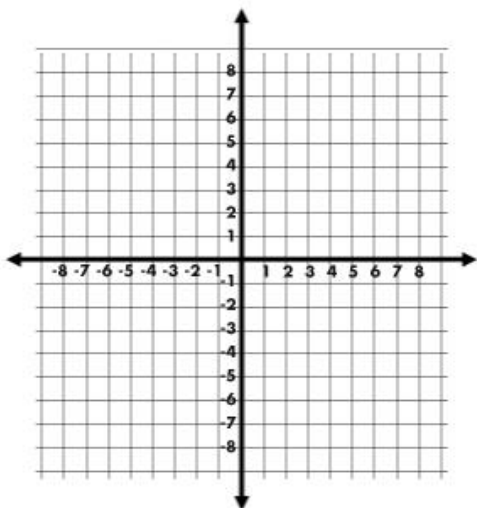
Since a is the x -intercept, we can plot that point $(3, 0)$ on the x -axis. Now draw a vertical line through that point in order to graph $x = 3$.



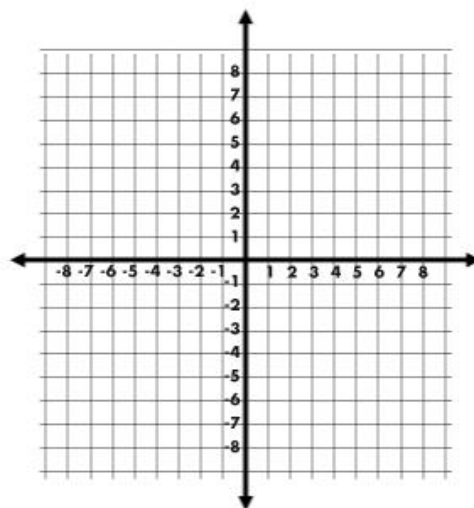
Vertical lines are *not* linear functions, as they do *not* pass the vertical line test. As a matter of fact, a vertical line drawn on top of an existing vertical line touches at infinitely many points, rather than at exactly one point. Vertical lines are the *only* lines that are not functions.

Problem 4.12

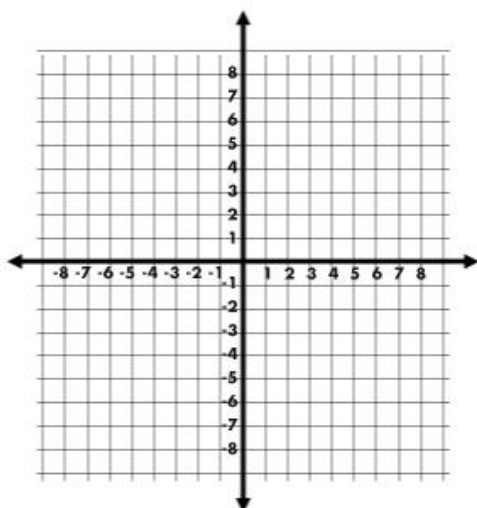
Graph $y = 5$ on the $x - y$ coordinate plane.

**Problem 4.13**

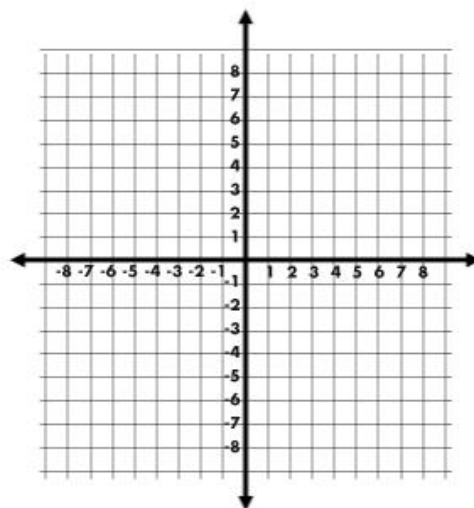
Graph $x = -6$ on the $x - y$ coordinate plane.

**Problem 4.14**

Graph $x = -5$ on the $x - y$ coordinate plane.

**Problem 4.15**

Graph $y = 1$ on the $x - y$ coordinate plane.



Section 5: Finding the Equations of Lines

When we are given information about a line, we are able to determine an equation for that line.

In the **slope-intercept form**, $y = mx + b$, there are 4 different variables. If we know 3 out of the 4 variables, then we can solve for the 4th variable and then determine the equation of the line.

In the **point-slope form**, $y - y_1 = m(x - x_1)$, we need to know the slope of the line, as well as the x and y coordinates of one point.

Problem 5.1

Write an equation of the line that has a slope of 5 and passes through the point $(2, -3)$.

Solution

We can write the **slope-intercept form** of the line by identifying all of the values that we know:

$m = 5$, $x = 2$, and $y = -3$

$$y = mx + b$$

Slope-intercept form of the equation of a line.

$$-3 = (5)(2) + b$$

Substitute in known values. The only unknown value in this case is b .

$$-3 = 10 + b$$

$$-3 - 10 = 10 - 10 + b$$

Solve for b .

$$-13 = b$$

$$y = 5x - 13$$

Now write the equation of the line using the slope, m and y -intercept, b .

Alternatively, we can write the **point-slope form** of the line as follows:

Again, we know that in this case $m = 5$. Also, for this problem we are only given one point, so we can call $x_1 = 2$, and $y_1 = -3$

$$y - y_1 = m(x - x_1)$$

Point-slope form of the equation of a line.

$$y - (-3) = 5(x - 2)$$

Substitute in known values.

$$y + 3 = 5(x - 2)$$

Simplify, if necessary. This is the point-slope form of the line.

Problem 5.2

Write an equation of the line that has a slope of -2 and passes through the point $(-3, 4)$.

(Use either method shown in Problem 5.1)

Sometimes we are given even less information about a line. Instead of being given the slope and a point on the line, we are only given two points. If this is the case then we must first use the slope formula to calculate the slope of the line.

Problem 5.3

Write an equation of the line that passes through the points $(-2, 4)$ and $(2, -8)$.

Solution

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-8 - 4}{2 - (-2)} = \frac{-12}{4} = -3$$

First, calculate the slope using the slope formula. We now know that $m = -3$.

$$\begin{aligned} y &= mx + b \\ 4 &= (-3)(-2) + b \\ 4 &= 6 + b \\ 4 - 6 &= 6 - 6 + b \\ -2 &= b \end{aligned}$$

Choose either of the original points given in the problem to be (x, y) . In this case, let's say that $(x, y) = (-2, 4)$ and substitute these values into the slope-intercept form, along with the slope, $m = -3$.

IMPORTANT: *Either point will work and give you the same equation.*

$$y = -3x - 2$$

Write the equation in slope-intercept form.

NOTE: Keep in mind that once you determine the slope of the line you can also use the **point-slope form** of the equation of the line as modeled in Problem 5.1. This follows for all problems in this section.

Problem 5.4

Write an equation of the line that passes through the points $(-1, 4)$ and $(-2, -2)$.

In summary, we have learned about the following different forms of the equations of a line:

Standard form: $Ax + By = C$

Slope-intercept form: $y = mx + b$

Point-slope form: $y - y_1 = m(x - x_1)$

Horizontal line: $y = b$

Vertical line: $x = a$

Parallel and Perpendicular Lines

The hallmark of **parallel lines** is that they do not intersect. This can only happen when the lines have the exact **same slope**. Alternatively, we can say that if the lines have the same slope, then they must be parallel.

Writing the Equation of a Line Parallel to a Given Line

Problem 5.5

Write an equation of the line that is parallel to $3x - 2y = 5$ and passes through the point $(6, -2)$.

Solution

$$\begin{aligned} 3x - 2y &= 5 \\ 3x - 3x - 2y &= -3x + 5 \\ -2y &= -3x + 5 \\ \frac{-2y}{-2} &= \frac{-3x}{-2} + \frac{5}{-2} \\ y &= \frac{3}{2}x - \frac{5}{2} \end{aligned}$$

We must first find the slope of the given line by solving for y .

The slope of the line is: $m = \frac{3}{2}$. This means that any line parallel to this line has the same slope.

$$\begin{aligned} y &= mx + b \\ -2 &= \left(\frac{3}{2}\right)(6) + b \\ -2 &= 9 + b \\ -2 - 9 &= 9 - 9 + b \\ -11 &= b \end{aligned}$$

Since we know the slope, $m = \frac{3}{2}$, we now only need to substitute in the values for $(x, y) = (6, -2)$ and then solve for b .

$$y = \frac{3}{2}x - 11$$

Now we can write the equation of the line that is parallel to $3x - 2y = 5$ and passes through the point $(6, -2)$.

Alternate solution

Once the slope is obtained, we can use the point-slope form of the line. Ultimately, we will express our solution in slope-intercept form.

$$y - y_1 = m(x - x_1)$$

Point-slope form of the equation of a line.

$$y - (-2) = \frac{3}{2}(x - 6)$$

Substitute the known values,
 $x = 6$, $y = -2$, $m = \frac{3}{2}$

$$y + 2 = \frac{3}{2}x - 9$$

Simplify by distributing.

$$y = \frac{3}{2}x - 11$$

Subtract 2 to solve for y .

Perpendicular lines are lines that intersect at a right angle (90°). There is a relationship between the slopes of perpendicular lines. They are said to be *negative reciprocals* of each other. This means that when we multiply the two slopes together they equal -1 .

NOTE: One exception to this fact is the case of horizontal and vertical lines. A horizontal and a vertical line will intersect at a right angle, however, the product of their slopes is not equal to -1 . Recall from Section 4 that the slope of a horizontal line is 0 and the slope of a vertical line is undefined.

There is an easy three-step process to determining the **slope of a perpendicular line**:

1. Identify the slope of the original line, and express it as a fraction.
2. Write the reciprocal (*flip numerator and denominator*).
3. Change the sign of the slope.

Problem 5.6

Find the slope of a line that is perpendicular to the line $y = -2x + 3$.

Solution

Original slope: $m = -2 = -\frac{2}{1}$

Reciprocal: $-\frac{1}{2}$.

Change the sign: $\frac{1}{2}$.

So the perpendicular slope is: $m = \frac{1}{2}$.

You can always check your answer by seeing if the two slopes multiplied together will equal -1 .

$$(-2)\left(\frac{1}{2}\right) = -1 \checkmark$$

Problem 5.7

Find the slope of a line that is *perpendicular* to the line $y = \frac{4}{5}x - 7$

Problem 5.8

Find the slope of a line that is *parallel* to the line $3x + 4y = 9$.

Writing the Equation of a Line that is Perpendicular to a Given Line

Problem 5.9

Write an equation of the line that is perpendicular to $-5x + y = 4$ and passes through the point $(-10, 7)$.

Solution

First we need to find the perpendicular slope. In this case we must solve for y in order to know the slope of the given line:

$$\begin{aligned} -5x + y &= 4 \\ -5x + \mathbf{5x} + y &= \mathbf{+5x} + 4 \\ y &= 5x + 4 \end{aligned}$$

So the slope of the original line is: $m = 5$.

Now to determine the perpendicular slope:

Original slope: $m = 5 = \frac{5}{1}$

Reciprocal: $\frac{1}{5}$.

Change the sign: $-\frac{1}{5}$

So the perpendicular slope is: $m = -\frac{1}{5}$

$$\begin{aligned} y &= mx + b \\ 7 &= -\frac{1}{5}(-10) + b \\ 7 - 2 &= 2 - 2 + b \\ 5 &= b \end{aligned}$$

$$y = -\frac{1}{5}x + 5$$

Since we now know the slope, $m = -\frac{1}{5}$ and one point on the line $(-10, 7)$, we can substitute these values into the slope-intercept form and solve for b .

This is the slope-intercept form of the line that is perpendicular to $-5x + y = 4$ and passes through the point $(-10, 7)$.

Alternate solution

Once the slope is obtained, we can use the point-slope form of the line. Ultimately, we will express our solution in slope-intercept form.

$$y - y_1 = m(x - x_1)$$

Point-slope form of the equation of a line.

$$y - 7 = -\frac{1}{5}(x - (-10))$$

Substitute the known values,
 $x = -10$, $y = 7$, $m = -\frac{1}{5}$

$$y - 7 = -\frac{1}{5}x - 2$$

Simplify by distributing.

$$y = -\frac{1}{5}x + 5$$

Add 7 to solve for y .

Problem 5.10

Write an equation of the line that is *perpendicular* to $3x + 2y = 4$ and passes through the point $(6, -8)$.

Problem 5.11

Write an equation of the line that is *parallel* to $2x + 4y = 7$ and passes through the point $(-4, 3)$.

Problem 5.12

Write an equation of the line that is *perpendicular* to $-6x - 3y = 5$ and passes through the point $(-6, 5)$.

Problem 5.13

Write an equation of the line that is *parallel* to $2x + 3y = 9$ and passes through the point $(3, -5)$.

Section 6: Solving Systems of Equations Graphically

In Section 4, we learned how to sketch the graph of a given line. Now, we will apply this skill to solving a system of equations in two variables.

We have already seen how to solve a linear equation in one variable in Section 1. Now, we will be faced with the question of solving two equations, each with two variables (usually x and y). This is an example of **a system of equations**.

We consider the following 3 scenarios when solving a system of linear equations:

The two lines intersect at one unique point

The two lines are parallel and therefore never intersect

The two lines represent the same line

Problem 6.1

Solve the following system of equations graphically. Check your solution.

$$\begin{aligned} 2x + 3y &= 9 \\ x - y &= 2 \end{aligned}$$

Solution

Notice that each of these equations represents a straight line in standard form. For each line, we will first convert the equation into slope-intercept form as below:

$$\begin{aligned} 2x + 3y &= 9 \\ 2x - 2x + 3y &= -2x + 9 \\ 3y &= -2x + 9 \\ \frac{3y}{3} &= \frac{-2x}{3} + \frac{9}{3} \\ y &= -\frac{2}{3}x + 3 \end{aligned}$$

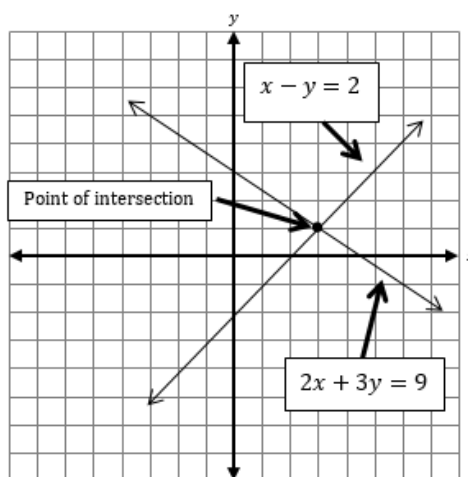
$$\begin{aligned} x - y &= 2 \\ x - x - y &= -x + 2 \\ -y &= -x + 2 \\ \frac{-y}{-1} &= \frac{-x}{-1} + \frac{2}{-1} \\ y &= x - 2 \end{aligned}$$

Now, we graph both lines on the *same* coordinate plane.

We check to see the point of intersection. Here, the lines intersect at $(3,1)$.

Therefore, our solution from the graph is $(x, y) = (3, 1)$.

We check our solution by substituting these values in each original equation.



$$\begin{aligned}
 2x + 3y &= 9 \\
 2(3) + 3(1) &\neq 9 \\
 6 + 3 &\neq 9 \\
 9 &= 9 \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 x - y &= 2 \\
 3 - 1 &\neq 2 \\
 2 &= 2 \quad \checkmark
 \end{aligned}$$

Since both equations are satisfied, we have verified that our solution (3,1) is correct!

(Note that both equations must be satisfied. Even if one equation fails to be satisfied, the point cannot be a solution!)

Since there was just one point of intersection, our solution is called a **unique solution** and our system of equations is called **consistent**.

Problem 6.2

Solve the following system of equations graphically. Check your solution.

$$\begin{aligned}
 2x - 3y &= 6 \\
 4x - 6y &= -6
 \end{aligned}$$

Solution

Once again, for each line, we will first convert the equation into slope-intercept form as below:

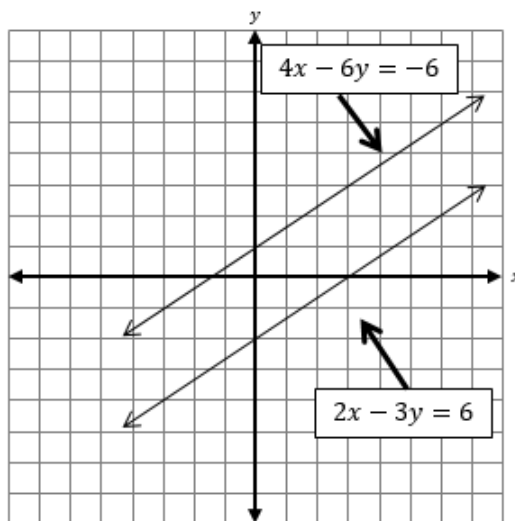
$$\begin{aligned}
 2x - 3y &= 6 \\
 2x - \mathbf{2x} - 3y &= -\mathbf{2x} + 6 \\
 -3y &= -2x + 6 \\
 \frac{-3y}{-3} &= \frac{-2x}{-3} + \frac{6}{-3} \\
 y &= \frac{2}{3}x - 2
 \end{aligned}$$

$$\begin{aligned}
 4x - 6y &= -6 \\
 4x - \mathbf{4x} - 6y &= -\mathbf{4x} - 6 \\
 -6y &= -4x - 6 \\
 \frac{-6y}{-6} &= \frac{-4x}{-6} - \frac{6}{-6} \\
 y &= \frac{2}{3}x + 1
 \end{aligned}$$

Now, we graph both lines on the *same* coordinate plane.

Notice that both lines have the same slope, that is $\frac{2}{3}$, and different y-intercepts. This means that the lines are parallel and they will **never intersect!**

This system of equations has **no solution**. We say that this system of equations is **inconsistent**.



Problem 6.3

Solve the following system of equations graphically. Check your solution.

$$\begin{aligned} 2x + 4y &= 8 \\ x + 2y &= 4 \end{aligned}$$

Solution

Once again, for each line, we will first convert the equation into slope-intercept form as below:

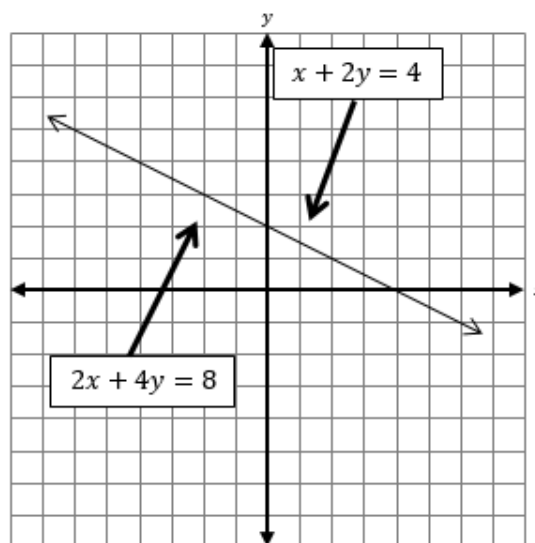
$$\begin{aligned} 2x + 4y &= 8 \\ 2x - 2x + 4y &= -2x + 8 \\ 4y &= -2x + 8 \\ \frac{4y}{4} &= \frac{-2x}{4} + \frac{8}{4} \\ y &= -\frac{1}{2}x + 2 \end{aligned}$$

$$\begin{aligned} x + 2y &= 4 \\ x - x + 2y &= -x + 4 \\ 2y &= -x + 4 \\ \frac{2y}{2} &= \frac{-x}{2} + \frac{4}{2} \\ y &= -\frac{1}{2}x + 2 \end{aligned}$$

Now, we graph both lines on the *same* coordinate plane.

Even though we set out to graph two separate lines, we ended up with just one line! This means that the two lines actually overlap. They touch at infinitely many points.

This system of equations has ***Infinitely many solutions***. We say that this system of equations is ***dependent***.



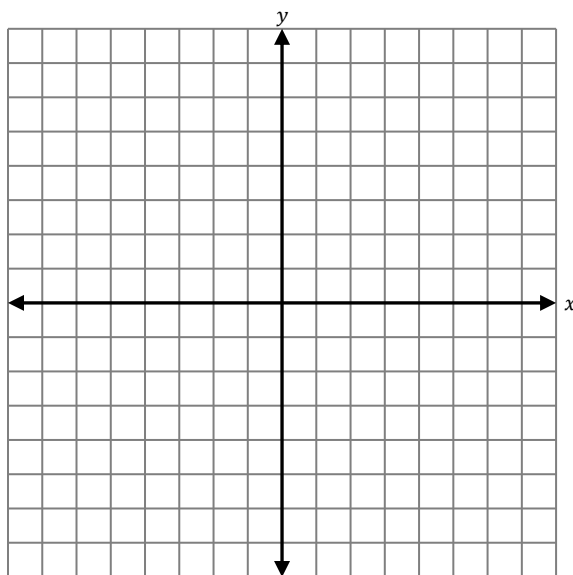
The solution can also be described as the set of all points that lie on the line $y = -\frac{1}{2}x + 2$.

In set notation we write, $\left\{ (x, y) \mid y = -\frac{1}{2}x + 2 \right\}$

Problem 6.4

Solve the following system of equations graphically. Check your solution.

$$\begin{aligned} 3x + 2y &= 10 \\ -x + y &= -5 \end{aligned}$$

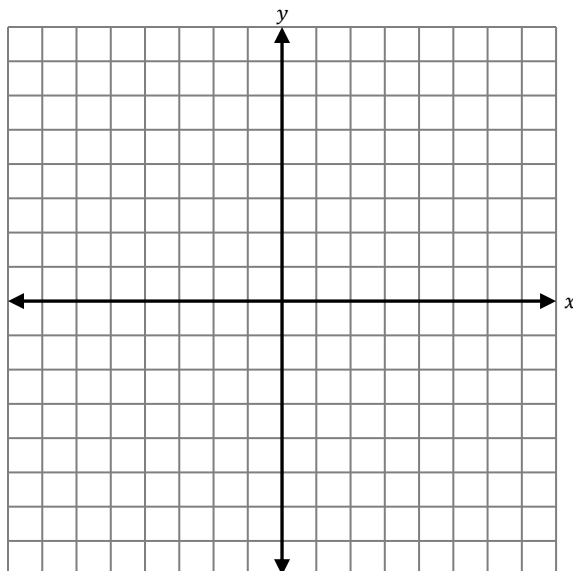


Problem 6.5

Solve the following system of equations graphically. Check your solution.

$$2x - 3y = -18$$

$$2x + 3y = 6$$

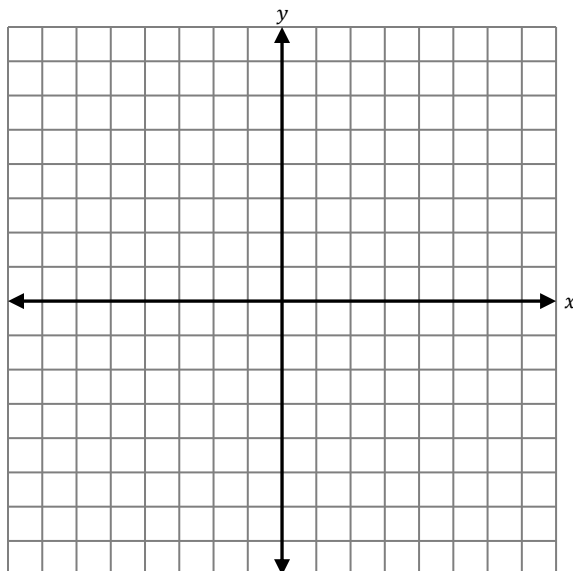


Problem 6.6

Solve the following system of equations graphically. Check your solution.

$$4x + 2y = 12$$

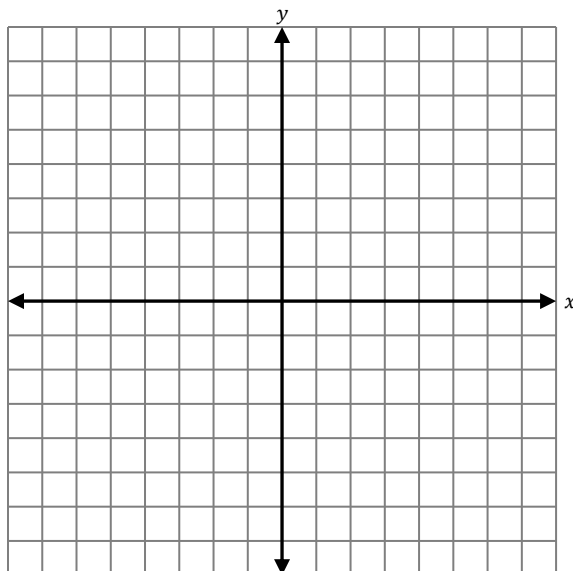
$$2x + y = 3$$



Problem 6.7

Solve the following system of equations graphically. Check your solution.

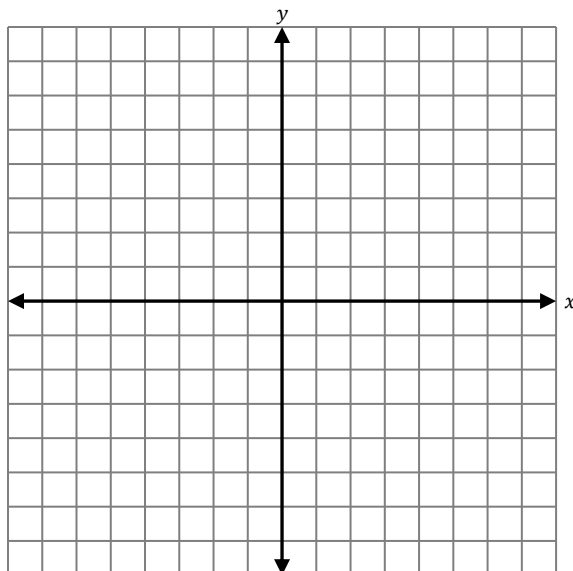
$$\begin{aligned}x + 3y &= -3 \\ 3x + 9y &= -9\end{aligned}$$



Problem 6.8

Solve the following system of equations graphically. Check your solution.

$$\begin{aligned}4x + 2y &= 14 \\ -3x + y &= -3\end{aligned}$$



Section 7: Solving Systems of Equations Algebraically

Now, we turn our attention to solving systems of equations using the **Addition Method**. Here, instead of graphing, we will manipulate the values of the coefficients to eliminate one of the variables. We see this below:

Problem 7.1

Solve the following system of equations algebraically. Check your solution.

$$4x + 5y = 24 \quad (\text{Eq. 1})$$

$$6x + 7y = 34 \quad (\text{Eq. 2})$$

Goal 1: Eliminate one variable, then solve for the remaining variable.

We will accomplish our goal of eliminating one variable by multiplying each equation by a (nonzero) integer so that the coefficients of the x 's are *opposites*. In Eq. 1 the coefficient of x is 4. In Eq. 2, the coefficient of x is 6. The (least) common multiple of 4 and 6 is 12. If we multiply 4 by -3 , the result will be -12 . If we multiply 6 by 2, the result will be 12. Let's proceed to see how this helps.

We start by multiplying every term in Eq. 1 by -3 .

$$4x + 5y = 24 \quad (\text{Eq. 1})$$

$$(-3)(4x) + (-3)(5y) = (-3)(24)$$

$$-12x - 15y = -72 \quad (\text{Eq. 3})$$

This gives us a new version of Eq. 1 which we have labeled Eq. 3. Notice that the coefficient of x is now -12 . We now multiply every term in Eq. 2 by 2.

$$6x + 7y = 34 \quad (\text{Eq. 2})$$

$$(2)(6x) + (2)(7y) = (2)(34)$$

$$12x + 14y = 68 \quad (\text{Eq. 4})$$

This gives us a new version of Eq. 2 which we have labeled Eq. 4. Notice that the coefficient of x is now 12. We will now add Eq. 3. and Eq. 4.

$$-12x - 15y = -72 \quad (\text{Eq. 3})$$

$$+ \quad 12x + 14y = 68 \quad (\text{Eq. 4})$$

$$\hline -y = -4 \quad (\text{Eq. 5})$$

$$y = 4$$

Notice that Eq. 5 (which we attained by adding Eq. 3 and Eq. 4) has only one variable and we were able to easily solve it to determine $y = 4$.

Goal 2: Find the other variable.

Now that we know the value of y , we substitute $y = 4$ into *either* of our first two *original equations*. We will demonstrate how you can substitute into either equation, but ordinarily you only need to do one or the other.

Substitute $y = 4$ into Eq. 1

$$4x + 5y = 24 \quad (\text{Eq. 1})$$

$$4x + 5(4) = 24$$

$$4x + 20 = 24 \quad (\text{Eq. 6})$$

$$4x = 4$$

$$x = 1$$

Substitute $y = 4$ into Eq. 2

$$6x + 7y = 34 \quad (\text{Eq. 2})$$

$$6x + 7(4) = 34$$

$$6x + 28 = 34 \quad (\text{Eq. 7})$$

$$6x = 6$$

$$x = 1$$

Once we substituted $y = 4$ into Eq. 1, the new Eq. 6. had only x , and it was easy to solve to see that $x = 1$. Similarly, we were able to substitute $y = 4$ into Eq. 2 and get Eq 7 with only x , and it was easy to solve to see $x = 1$.

At this point, we hope we have reached our solution $x = 1, y = 4$ which we write as an ordered pair $(1,4)$. But before moving on, we will check that we have the correct solution by substituting $(1,4)$ into ***both original equations***.

Goal 3: Check the answer is correct**Check**

Substitute $(1,4)$ into Eq. 1.

$$4x + 5y = 24$$

$$4(1) + 5(4) \neq 24$$

$$4 + 20 \neq 24$$

$$24 = 24 \quad \checkmark$$

Substitute $(1,4)$ into Eq. 2.

$$6x + 7y = 34$$

$$6(1) + 7(4) \neq 34$$

$$6 + 28 \neq 34$$

$$34 = 34 \quad \checkmark$$

Our solution is correct. The solution is $(1,4)$.



Problem 7.2

Solve the following system of equations algebraically. Check your solution.

$$\begin{aligned}-3x + 5y &= 19 \\ 9x - 2y &= 8\end{aligned}$$

Problem 7.3

Solve the following system of equations algebraically. Check your solution.

$$\begin{aligned}2x + y &= -1 \\ 7x - 3y &= -23\end{aligned}$$

Problem 7.4

Solve the following system of equations algebraically. Check your solution.

$$\begin{aligned} -4x + 9y &= 9 \\ x - 3y &= -6 \end{aligned}$$

Problem 7.5

Solve the following system of equations algebraically. Check your solution.

$$\begin{aligned} -3x + 10y &= 25 \\ 2x - 7y &= -18 \end{aligned}$$

Now let's take a look what happens when we solve algebraically and the result is either the two lines are parallel (no solution) or the two lines are the same line (infinitely many solutions)

Problem 7.6

Solve the following system of equations algebraically. Check your solution.

$$4x + 6y = 22 \quad (\text{Eq. 1})$$

$$2x + 3y = 5 \quad (\text{Eq. 2})$$

We start by multiplying every term in Eq. 2 by -2 .

$$2x + 3y = 5 \quad (\text{Eq. 2})$$

$$(-2)(2x) + (-2)(3y) = (-2)(5)$$

$$-4x - 6y = -10 \quad (\text{Eq. 3})$$

This gives us a new version of Eq. 2 which we have labeled Eq. 3. We don't have to multiply Eq. 1 by anything.

We will now add Eq. 1. to Eq. 3.

$$\begin{array}{rcl} 4x + 6y & = & 22 \quad (\text{Eq. 1}) \\ + \quad -4x - 6y & = & -10 \quad (\text{Eq. 3}) \\ \hline 0 & \neq & 12 \end{array}$$

Something different happened here. Again, both variables were eliminated! The bad news is we are left with a false statement since 0 does not equal 12. Therefore, there is **no solution**.

Let's see what happens if we try the graphical method of solving this system. We start by putting each equation into slope-intercept form ($y = mx + b$).

$$\begin{aligned} 4x + 6y &= 22 \\ 4x - 4x + 6y &= -4x + 22 \\ 6y &= -4x + 22 \\ \frac{6y}{6} &= \frac{-4x}{6} + \frac{22}{6} \\ y &= -\frac{2}{3}x + \frac{11}{3} \end{aligned}$$

$$\begin{aligned} 2x + 3y &= 5 \\ 2x - 2x + 3y &= -2x + 5 \\ 3y &= -2x + 5 \\ \frac{3y}{3} &= \frac{-2x}{3} + \frac{5}{3} \\ y &= -\frac{2}{3}x + \frac{5}{3} \end{aligned}$$

We see that both lines have the *same slope* **but** they have *different y-intercepts*. We have seen this before. Recall, this was the situation when we have two *parallel* lines. In this case, there is *no solution*, and the system is said to be *inconsistent*.

■

Problem 7.7

Solve the following system of equations algebraically. Check your solution.

$$2x + 3y = 11 \quad (\text{Eq. 1})$$

$$4x + 6y = 22 \quad (\text{Eq. 2})$$

We start by multiplying every term in Eq. 1 by -2 .

$$2x + 3y = 11 \quad (\text{Eq. 1})$$

$$(-2)(2x) + (-2)(3y) = (-2)(11)$$

$$-4x - 6y = -22 \quad (\text{Eq. 3})$$

This gives us a new version of Eq. 1 which we have labeled Eq. 3. We don't have to multiply Eq. 2 by anything (or you can say we multiplied by 1 which changes nothing).

We will now add Eq. 3. and Eq. 2.

$$\begin{array}{r} -4x - 6y = -22 \quad (\text{Eq. 3}) \\ + \quad 4x + 6y = 22 \quad (\text{Eq. 2}) \\ \hline 0 = 0 \end{array}$$

Something different happened here. **Both** variables were eliminated! The good news is we are left with an equality $0 = 0$ which is a **true** statement. Therefore, there are **infinitely many solutions**.

Let's see what happens if we try the graphical method of solving this system. We start by putting each equation into slope-intercept form ($y = mx + b$).

$$\begin{array}{l} 2x + 3y = 11 \\ 2x - 2x + 3y = -2x + 11 \\ 3y = -2x + 11 \\ \frac{3y}{3} = \frac{-2x}{3} + \frac{11}{3} \\ y = -\frac{2}{3}x + \frac{11}{3} \end{array}$$

$$\begin{array}{l} 4x + 6y = 22 \\ 4x - 4x + 6y = -4x + 22 \\ 6y = -4x + 22 \\ \frac{6y}{6} = \frac{-4x}{6} + \frac{22}{6} \\ y = -\frac{2}{3}x + \frac{11}{3} \end{array}$$

We see that both lines have the **same slope** **and** the **same y-intercept**. We have seen this before. Recall this was the situation when we actually have only one line instead of two. In this case, there are **infinitely many solutions**, and the system is said to be **dependent**.

The solution can also be described as the set of all points that lie on the line $y = -\frac{2}{3}x + \frac{11}{3}$.

In set notation we write, $\left\{ (x, y) \mid y = -\frac{2}{3}x + \frac{11}{3} \right\}$

■

Problem 7.8

Solve the following system of equations algebraically. Check your solution.

$$15x + 12y = 10$$

$$5x + 4y = 3$$

Problem 7.9

Solve the following system of equations algebraically. Check your solution.

$$4x + 3y = 1$$

$$12x + 9y = 3$$

Section 8: Integral Exponents

We will begin our discussion of integral exponents by breaking down several rules. The Product Rule, the Quotient Rule and the Power Rule are three basic rules that will assist us in solving more complicated problems involving exponents.

First recall the parts of an exponential expression:

The BASE (x) is the value being multiplied

$x^4 = x \cdot x \cdot x \cdot x$

The EXPONENT indicates how many times to multiply the base.

The Product Rule

Problem 8.1

Multiply.

$$(x^4)(x^2)$$

Solution

To understand the product rule, it is helpful to expand each expression.

$$(x^4)(x^2) = \underbrace{(x \cdot x \cdot x \cdot x)(x \cdot x)}_{\text{6 factors of } x}$$

$$(x^4)(x^2) = x^6$$

Product Rule

When multiplying two (or more) exponential expressions with the same base, keep the base and **ADD** the exponents.

$$x^m \cdot x^n = x^{m+n}$$

When coefficients are involved, simply multiply the coefficients and add the exponents of the like bases.

Problem 8.2

Multiply.

$$(-2a^3b^2c^4)(5ab^3)$$

Solution

Since multiplication is commutative, we can rearrange factors to more easily apply the product rule.

$$(-2 \cdot 5)(a^3 \cdot a)(b^2 \cdot b^3)(c^4) = -10a^4b^5c^4$$

The Quotient Rule

Problem 8.3

Simplify.

$$\frac{x^6}{x^2}$$

Solution

Again, it is helpful to expand the expressions.

$$\frac{x^6}{x^2} = \frac{\overbrace{x \cdot x \cdot x \cdot x \cdot \cancel{x} \cdot \cancel{x}}^{\cancel{x} \cdot \cancel{x}}}{\cancel{x} \cdot \cancel{x}} = x^4$$

Cancel factors that appear in the numerator and denominator

Notice that the exponent in the numerator is larger. Therefore, after simplifying, the expression remains in the numerator.

When coefficients are involved, simply divide or simplify the fraction, and subtract the exponents with like bases.

Problem 8.4

Simplify.

$$\frac{-18m^6n^3}{3m^4n}$$

Solution

Divide coefficients and subtract exponent of the factors that have the same base

$$\frac{-18m^6n^3}{3m^4n} = -6m^2n^2$$

In some expressions, there may be more factors in the denominator. In this case, after simplifying, the expression remains in the denominator.

Problem 8.5

Simplify.

$$\frac{x^3}{x^5}$$

SolutionBy expanding the exponents, we see that there are more x 's in the denominator.

$$\frac{x^3}{x^5} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x}}{x \cdot x \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x}} = \frac{1}{x^2}$$

Quotient Rule

When dividing two exponential expressions with the same base, keep the base and **SUBTRACT** the exponents. (Assume the denominator $\neq 0$)

$$\frac{x^m}{x^n} = x^{m-n} \text{ when } m > n \quad \text{and} \quad \frac{x^m}{x^n} = \frac{1}{x^{n-m}} \text{ when } n > m$$

The Zero Power RuleThe Quotient Rule also helps us to understand **The Zero Power Rule**.**Problem 8.6**

Simplify.

$$\frac{x^3}{x^3}$$

Solution

We will take a look at this expression in two different ways:

$$\left. \begin{array}{ll} \frac{x^3}{x^3} = 1 & \text{Any value divided by itself is 1} \\ \frac{x^3}{x^3} = x^0 & \text{Quotient Rule (Subtract exponents)} \end{array} \right\} \text{Therefore, } x^0 = 1$$

Zero Power Rule

Any base (except 0) raised to the 0 power is 1.

$$x^0 = 1 \quad \text{and} \quad \frac{x^m}{x^m} = 1, \quad x \neq 0$$

Now let's practice using the Product, Quotient and Zero Power rules:

Problem 8.7

Multiply. $(3xy)(x^2y^8)$

Problem 8.8

Multiply. $(-4a^0b^4c^2)(-2a^2bc^3)$

Problem 8.9

Simplify. $\frac{a^7b^5}{a^4b}$

Problem 8.10

Simplify. $\frac{-20x^4y^2}{4x^9}$

Problem 8.11

Simplify. $\frac{7p^8q}{14p^4q^3}$

Problem 8.12

Simplify. $\frac{(x^5y)(x^2y)}{x^9y^2}$

Negative Exponents

Next we'll be taking a look at negative exponents. At times we will be presented with an expression that already contains negative exponents and, at other times, the order in which we choose to subtract may lead to negative exponents.

We can see an example of negative exponents by applying the Quotient Rule in both directions.

Problem 8.13

Simplify. $\frac{x^3}{x^5}$

Solution

We already looked into this example and saw that there is more factors of x in the denominator. Therefore, we subtracted **the numerator FROM the denominator** and the remaining x 's stayed in the denominator.

$$\frac{x^3}{x^5} = \frac{1}{x^{5-3}} = \frac{1}{x^2}$$

Now let's apply the Quotient Rule by subtracting **the denominator from the numerator**.

$$\frac{x^3}{x^5} = x^{3-5} = x^{-2}$$

We have now shown that $x^{-2} = \frac{1}{x^2}$

In a similar way, we can deal with negative exponents in the denominator by moving them to the numerator.

Negative Exponent Rule

For any nonzero base,

$$x^{-n} = \frac{1}{x^n} \quad \text{and} \quad \frac{1}{x^{-n}} = x^n$$

Exponential expressions should always be expressed with positive exponents only. Negative exponents can be eliminated by moving the factor from the numerator to the denominator, or the denominator to the numerator, and changing the exponent to its positive.

Problem 8.14

Simplify. Express with positive exponents only.

$$\frac{-2x^{-3}y^2}{z^{-5}}$$

SolutionThe factors that contain a negative exponent are x and z .**NOTE:** The coefficient does NOT contain a negative exponent. It is simply a negative number. Therefore, it will remain in the numerator.

$$\frac{-2x^{-3}y^2}{z^{-5}} = \frac{-2y^2z^5}{x^3}$$

Problem 8.15

Simplify. Express with positive exponents only.

$$\frac{x^{-2}}{5}$$

Problem 8.16

Simplify. Express with positive exponents only.

$$\frac{-8x^5}{2y^{-4}}$$

Problem 8.17

Simplify. Express with positive exponents only.

$$\frac{a^{-1}b^{-7}}{c^{-3}}$$

Problem 8.18

Simplify. Express with positive exponents only.

$$\frac{6m^{-3}n^4}{12m^3n^{-2}}$$

The Power Raised to a Power Rule (or The Power Rule)

Problem 8.19

Simplify.

$$(x^2)^3$$

Solution

To simplify a power raised to a power, we think of raising x^2 to the third power.

$$(x^2)^3 = x^2 \cdot x^2 \cdot x^2 = x^6$$

The Power Rule

When raising a power to a power, keep the base and **MULTIPLY** the exponents.

$$(x^m)^n = x^{m \cdot n}$$

The last two rules we will cover involve a product or quotient inside of parentheses being raised to a power. In both of these cases we can think of “distributing” the power to each and every factor inside the parentheses.

Product Raised to a Power Rule

Problem 8.20

Simplify. Express with positive exponents only.

$$(3x^2y^4)^3$$

Solution

In this expression, each factor of $(3x^2y^4)$ is being multiplied by itself 3 times. Working step by step leads us to “distribution” of the power of 3 to each factor.

$$\begin{aligned} (3x^2y^4)^3 &= (3x^2y^4) \cdot (3x^2y^4) \cdot (3x^2y^4) \\ &= (3 \cdot 3 \cdot 3) \cdot (x^2 \cdot x^2 \cdot x^2) \cdot (y^4 \cdot y^4 \cdot y^4) \\ &= (3)^3 \cdot (x^2)^3 \cdot (y^4)^3 \\ &= 27x^6y^{12} \end{aligned}$$

Multiply the expression by itself three times

Rearrange

Notice that the exponent has been “distributed”.

Simplify

Product Raised to a Power Rule

Raise each factor in the parentheses to the power n .

$$(xy)^n = x^n y^n$$

Quotient Raised to a Power Rule

Problem 8.21

Simplify.

$$\left(\frac{5x^4}{y^3}\right)^2$$

Solution

Similar to products raised to a power, the parentheses tell us that $\frac{5x^4}{y^3}$ is being multiplied by itself 2 times. The power of 2 will be “distributed” to each factor in the numerator AND denominator.

$$\left(\frac{5x^4}{y^3}\right)^2 = \left(\frac{5x^4}{y^3}\right) \cdot \left(\frac{5x^4}{y^3}\right) = \frac{(5 \cdot 5) \cdot (x^4 \cdot x^4)}{y^3 \cdot y^3} = \frac{(5)^2 (x^4)^2}{(y^3)^2}$$

$$\text{Therefore, } \left(\frac{5x^4}{y^3}\right)^2 = \frac{25x^8}{y^6}$$

Quotient Raised to a Power Rule

Raise each factor in the numerator AND denominator to the power n .

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

In the remaining problems, we will combine all exponent rules. In general, an exponential expression is considered simplified when:

All parentheses are removed

No negative exponents or zero exponents exist

Each base appears only once

Coefficients are multiplied, divided or simplified

Problem 8.22

Simplify.

$$(-5m^4)^2$$

Problem 8.23

Simplify.

$$(x^4)(6xy)^2$$

Problem 8.29

Simplify. Express with positive exponents only.

$$\left(\frac{-2}{x^5}\right)^3$$

Problem 8.30

Simplify. Express with positive exponents only.

$$\left(\frac{x^{-2}}{y^{-4}}\right)^5$$

Problem 8.31

Simplify. Express with positive exponents only.

$$\left(\frac{4a^{-6}}{-8b^3}\right)^4$$

Problem 8.32

Simplify. Express with positive exponents only.

$$(4m^3n^{-4})^{-3}$$

Problem 8.33

Simplify. Express with positive exponents only.

$$(3s^7t^{-5})^2(4s^0t)$$

Problem 8.34

Simplify. Express with positive exponents only.

$$\left(\frac{5w^{-4}v^{-6}}{15x^3}\right)^{-3}$$

Section 9: Greatest Common Factors and Factoring by Grouping

Before we begin the topic of factoring, let's review how to perform the four operations (addition, subtraction, multiplication and division) on polynomials with one or more than one variable.

Adding or Subtracting Polynomials

To add or subtract polynomials, you first need to identify *like terms* and then combine them according to the indicated operation.

Problem 9.1

Add.

$$(6x^2 - 7xy + 5y) + (11x^2 + xy - 8y)$$

Solution

$$(6x^2 - 7xy + 5y) + (11x^2 + xy - 8y)$$

When adding, we may remove parentheses.

Carefully identify like terms.

$$6x^2 - 7xy + 5y + 11x^2 + xy - 8y$$

Rewrite with like terms next to each other. Remember to keep the sign of each term!

$$\underbrace{6x^2 + 11x^2} - \underbrace{7xy + xy} + \underbrace{5y - 8y}$$

Combine like terms.

$$17x^2 - 6xy - 3y$$

NOTE: Be careful! The parentheses are used to distinguish two separate polynomials, and **do not** indicate multiplication of polynomials.

Problem 9.2

Subtract.

$$(8x^2y + 13xy + 7) - (12x^2y - 4xy + 3)$$

Solution

$$(8x^2y + 13xy + 7) - (12x^2y - 4xy + 3)$$

Since there is a subtraction sign outside the second polynomial, we must *change the sign of each term in the second polynomial*.

$$8x^2y + 13xy + 7 - 12x^2y + 4xy - 3$$

Rewrite with like-terms next to each other.

$$\underbrace{8x^2y - 12x^2y} + \underbrace{13xy + 4xy} + \underbrace{7 - 3}$$

Combine like terms.

$$-4x^2y + 17xy + 4$$

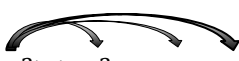
Multiplying Polynomials

In Section 8 we learned how to multiply two monomials together by multiplying the coefficients and adding the exponents of like bases (the Product Rule). When multiplying polynomials, we will need to use the distributive property.

Problem 9.3

Multiply.

$$(5x^2)(6x^3 - 3xy + 2)$$



$$(5x^2)(6x^3 - 3xy + 2)$$

Distribute $5x^2$ to each term in the parentheses

$$(5x^2)(6x^3) - (5x^2)(3xy) + (5x^2)(2)$$

Multiply coefficients and multiply variable using the Product Rule (add exponents)

$$30x^5 - 15x^3y + 10x^2$$

Problem 9.4

Multiply.

$$(3x - 6)(5x + 2)$$

Solution

When multiplying a binomial by another binomial, we need to approach multiplication differently. This type of problem will require us to distribute twice.

$$3x(5x + 2) - 6(5x + 2)$$

Distribute each term in the first binomial to the second binomial.

$$15x^2 + 6x - 30x - 12$$

Complete the distribution.

$$15x^2 - 24x - 12$$

Combine like terms.

Notice that we are able to combine the *middle terms* after we do this distribution.

When the binomials have terms that are exactly the same, but one has an addition sign and the other has a subtraction sign, then we say that these binomials are the *sum and difference of two terms*. Another name for these is *conjugates*. We can use the same methodology as above to multiply.

Problem 9.5

Multiply.

$$(2x + 5)(2x - 5)$$

Solution

$$2x(2x - 5) + 5(2x - 5)$$

Distribute each term in the first binomial with the second binomial.

$$= 4x^2 - 10x + 10x - 25$$

Complete the distribution.

$$= 4x^2 - 25$$

Combine like terms.

NOTE: This time the middle terms *cancelled each other out*. This will *always* be the case for the sum and difference of two terms. Since this is the case, you could take a shortcut and just multiply the first two terms of the binomials and the second two terms of the binomials.

$$(2x + 5)(2x - 5) = (2x)(2x) + (5)(-5) = 4x^2 - 25$$

It is important to note that $4x^2 - 25 = (2x)^2 - (5)^2$. This is known as a *difference of two squares*.

Dividing Polynomials

In Section 8, we learned how to divide two monomials by dividing the coefficients and subtracting the exponents in the denominator from those with like bases in the numerator (the Quotient Rule).

Problem 9.6

Divide.

$$(12x^6y^3 - 16x^4y^2 + 4xy) \div (4xy)$$

Solution

When dividing a polynomial by a monomial, it is best to rewrite the division in fraction form, and divide each term by the monomial and calculate what is left for each term, paying extra attention to signs.

$$\begin{aligned} & \frac{12x^6y^3 - 16x^4y^2 + 4xy}{4xy} \\ &= \frac{12x^6y^3}{4xy} - \frac{16x^4y^2}{4xy} + \frac{4xy}{4xy} \end{aligned}$$

Rewrite as a fraction.

Divide each term in the numerator by the monomial in the denominator.

$$= 3x^5y^2 - 4x^3y + 1$$

Simplify.

Problem 9.7

Add.

$$(-4x^3y^2 + 13xy - 9y) + (2x^3y^2 - 3xy)$$

Problem 9.8

Subtract.

$$(-2x^2y - 8xy - 5) - (-7x^2y - xy + 11)$$

Problem 9.9

Multiply.

$$(3xy^2)(-5x^2y^7 + 6xy^3 - 2y)$$

Problem 9.10

Multiply.

$$(3x + 6)(7x - 2)$$

Problem 9.11

Multiply.

$$(2a - 3b)(5a - 2b)$$

Problem 9.12

Multiply.

$$(5x + 3)(5x - 3)$$

Problem 9.13

Divide.

$$(14x^8y^6 - 21x^6y^4 + 7x^2y) \div (7x^2y)$$

Problem 9.14

Divide.

$$\frac{8x^5y^3 + 6x^3y^2 - 2xy}{2xy}$$

Factoring Out the Greatest Common Factor (GCF)

Whenever a problem involves factoring, our first step is to look for a *greatest common factor*. The **greatest common factor**, abbreviated GCF, is the greatest factor that divides all of the terms of the polynomial. We can find the GCF to factor a polynomial as follows:

First look at the coefficients of all the terms to find the greatest factor they have in common

Next, look at each of the variables that all of the terms have in common. The variable with the lowest exponent, or power, will be common to all the terms.

Once you have found the GCF, you should divide each term by the GCF to determine the remaining factor. *Include the GCF in your factorization*

Problem 9.15

Factor.

$$18x^3y^2 - 27x^2y$$

Solution

If we look at the coefficients, 18 and 27, we find that 9 is the greatest factor.

If we examine x^3 and x^2 , we see that x^2 has the lowest exponent. Between y^2 and y , y has the lowest exponent.

Therefore, $9x^2y$ is the GCF. Next, we divide each term by the GCF, to find the remaining factor.

$$\begin{aligned} &18x^3y^2 - 27x^2y \\ &9x^2y \left(\frac{18x^3y^2}{9x^2y} - \frac{27x^2y}{9x^2y} \right) \\ &= 9x^2y(2xy - 3) \end{aligned}$$

Important Note: If the leading coefficient of the polynomial is **negative**, then it is useful to factor out a common factor with a **negative coefficient**. The GCF is preceded by a *negative sign*.

Problem 9.16

Factor.

$$-9x^3y^2 + 6xy^3$$

Solution

In this case, since the polynomial has a negative leading coefficient, the GCF is $-3xy^2$.

$$\begin{aligned} &-9x^3y^2 + 6xy^3 \\ &-3xy^2 \left(\frac{-9x^3y^2}{-3xy^2} + \frac{6xy^3}{-3xy^2} \right) \\ &= -3xy^2(3x - 2y) \end{aligned}$$

Factoring Out a Greatest Binomial Factor

In the previous examples we were factoring a monomial, or single term. There are times that we need to factor out a GCF that is a binomial.

Problem 9.17

Factor.

$$4(x + 8) + 2y(x + 8)$$

Solution

Here we can see that each of the terms has $(x + 8)$ in common.

$$4(x + 8) + 2y(x + 8)$$

We can factor out $(x + 8)$ from each term and list what is remaining in a *separate set of parentheses*.

$$(x + 8)(4 + 2y)$$

Problem 9.18

Factor.

$$7y(x - 5) - (x - 5)$$

Solution

We can see that $(x - 5)$ is common to both terms. In this case, it is best to rewrite this problem in the following way, in order to help us factor.

$$7y(x - 5) - 1(x - 5)$$

Now we can proceed as above, by factoring out $(x - 5)$ and list what is remaining in a separate set of parentheses.

$$(x - 5)(7y - 1)$$

Problem 9.19

Factor.

$$14x^2y^5z + 21xy^3$$

Problem 9.20

Factor.

$$10x^4y^2 + 25x^2y^3 - 15x^3y$$

Problem 9.21

Factor.

$$-12x^2y^5 + 21xy^2$$

Problem 9.22

Factor.

$$-20x^2y^3 - 15x^3y$$

Problem 9.23

Factor.

$$5(a + 2b) + 3c(a + 2b)$$

Problem 9.24

Factor.

$$8b(2a + 7) - (2a + 7)$$

Factoring by Grouping

When we are asked to factor a polynomial with four terms, we will often accomplish this by using **factoring by grouping**. This means that we will group the first two terms together to find a GCF and then group the remaining two terms together to find a GCF. From this process, we hope to then find a common binomial factor.

Problem 9.25

Factor.

$$x^3 - 4x^2 + 6x - 24$$

Solution

$$\underbrace{x^3 - 4x^2} + \underbrace{6x - 24}$$

Group together terms that have common factors.

$$x^2(x - 4) + 6(x - 4)$$

Find the GCF of each group.

$$x^2(x - 4) + 6(x - 4)$$

Notice the common binomial factor $(x - 4)$.

$$(x - 4)(x^2 + 6)$$

Factor out the GCF.

NOTE: You can check the answer by multiplying the two binomial factors together. If factored correctly, you will get the original polynomial.

Problem 9.26

Factor.

$$3xy - 27y - 11x + 99$$

Solution

$$\underbrace{3xy - 27y} - \underbrace{11x + 99}$$

Group together terms that have common factors.

$$3y(x - 9) - 11(x - 9)$$

Find the GCF of each group. Be careful to factor out the **negative GCF** from the second group.

$$3y(x - 9) - 11(x - 9)$$

Notice the common binomial factor $(x - 9)$.

$$(x - 9)(3y - 11)$$

Factor out the GCF.

Problem 9.27Factor and *check*.

$$3x^3 + 2x^2 + 12x + 8$$

Problem 9.28Factor and *check*.

$$2x^2 - 4x - 3x + 6$$

Problem 9.29Factor and *check*.

$$y^2 - ay - by + ab$$

Problem 9.30Factor and *check*.

$$5a^2x + 10a^2y - 3bx - 6by$$

NOTE: Sometimes it may be necessary to rearrange the terms before you start to group them. Consider the following polynomial:

$$7x^2 - 4y + 14x - 2xy$$

The first two terms have no common factor other than 1. If we rearrange the terms and try factoring by grouping again, then we may have a better result:

$$7x^2 + 14x - 2xy - 4y$$

Now it is possible to find a common factor for first two terms as well as the second two terms:

$$7x(x + 2) - 2y(x + 2)$$

We can now proceed to factor out the GCF:

$$(x + 2)(7x - 2y)$$

Section 10: Factoring Trinomials

In this section we will factor trinomials in the form $ax^2 + bx + c$. When factoring a trinomial (or any polynomial) our goal is to write the original expression as a product of polynomials. More specifically, a trinomial will often factor into a binomial times a binomial.

In the previous section, we factored by grouping. We will use this method to factor trinomials. However, the grouping method requires four terms while a trinomial only has three terms! We will use the method of splitting the middle term to create four terms, and then factor by grouping.

Problem 10.1

Factor. Check your answer by multiplying.

$$5x^2 + 13x + 6$$

Solution

We will start by splitting the middle term ($+13x$) into two terms that have a sum of $+13x$ so that we have a polynomial with four terms. It will be helpful to first identify the values involved.

$$5x^2 + 13x + 6$$

$a = 5$ $b = 13$ $c = 6$

In order to find the correct pair of integers, we look for **factors of $a \cdot c$ that have a sum of b** .

In this case we need two numbers that multiply to $5 \cdot 6 = 30$ and add to 13.

Factors of 30	Sum
1 and 30	31
2 and 15	17
3 and 10	13
5 and 6	11

$$5x^2 + 13x + 6$$

$$5x^2 + 10x + 3x + 6$$

Now that we have four terms, we can use the grouping method. It is important to note that each of the new terms must include an x and an operation.

$$\begin{aligned}
 &5x^2 + 10x + 3x + 6 \\
 &(5x^2 + 10x) + (3x + 6) \\
 &5x(x + 2) + 3(x + 2) \\
 &(x + 2)(5x + 3)
 \end{aligned}$$

We can check by multiplying the resulting polynomials:

$$\begin{aligned}
 (x + 2)(5x + 3) &= 5x^2 + 3x + 10x + 6 \\
 &= 5x^2 + 13x + 6
 \end{aligned}$$

The same process is used when b and/or c are negative. However, we must take extra care when finding the correct pair of integers that satisfy **factors of $a \cdot c$ that have a sum of b** .

To **eliminate some combinations of factors**, keep the following in mind:

If the **product of two numbers is positive**, both factors must be positive OR both factors must be negative.

- If their sum is positive, then both factors must be positive
- If their sum is negative, then both factors must be negative

If the **product of two numbers is negative**, then one factor must be positive and one factor must be negative.

- If their sum is positive, then the factor with larger absolute value must be positive
- If their sum is negative, then the factor with larger absolute value must be negative

Problem 10.2

Factor. Check your answer by multiplying.

$$8x^2 - 11x + 3$$

$$a = 8 \quad b = -11 \quad c = 3$$

Solution

To split the middle term, we need factors of $8 \cdot 3 = 24$, that have a sum of -11 . Since the product is positive and the sum is negative, BOTH factors must be negative.

$$\begin{aligned} &8x^2 - 11x + 3 \\ &\quad \swarrow \quad \searrow \\ &8x^2 - 8x - 3x + 3 \\ &(8x^2 - 8x) - (3x + 3) \\ &8x(x - 1) - 3(x - 1) \\ &(x - 1)(8x - 3) \end{aligned}$$

Factors of 24	Sum
-1 and -24	-25
-2 and -12	-14
-3 and -8	-11
-4 and -6	-10

Check your answer by multiplying the binomials.

$$\begin{aligned} (x - 1)(8x - 3) &= 8x^2 - 3x - 8x + 3 \\ &= 8x^2 - 11x + 3 \end{aligned}$$

Problem 10.3

Factor. Check your answer by multiplying.

$$x^2 + 12x - 45$$

Solution

We are looking for factors of -45 that have a sum of $+12$. Since the product is negative and the sum is positive, the factor with the larger absolute value must be positive.

$$\begin{aligned} & x^2 + 12x - 45 \\ & \quad \swarrow \quad \searrow \\ & x^2 - 3x + 15x - 45 \\ & (x^2 - 3x) + (15x - 45) \\ & x(x - 3) + 15(x - 3) \\ & (x - 3)(x + 15) \end{aligned}$$

Factors of -45	Sum
-1 and 45	44
-3 and 15	12
-5 and 9	4

NOTE: When the leading coefficient is 1 (the coefficient of the x^2 term) the trinomial can be factored more easily. Once you identify which factor to use, the grouping process can be skipped and the factors can be placed directly into the binomials.

$$\begin{aligned} & x^2 + 12x - 45 \\ & (x - 3)(x + 15) \end{aligned}$$

Factors of -45	Sum
-1 and 45	44
-3 and 15	12
-5 and 9	4

Problem 10.4

Factor. Check your answer by multiplying.

$$4x^2 - 3x - 10$$

Solution

We are looking for factors of -40 that have a sum of -3 . Since the product is negative and the sum is negative, the factor with the larger absolute value must be negative.

$$\begin{aligned} & 4x^2 - 3x - 10 \\ & \quad \swarrow \quad \searrow \\ & 4x^2 - 8x + 5x - 10 \\ & (4x^2 - 8x) + (5x - 10) \\ & 4x(x - 2) + 5(x - 2) \\ & (x - 2)(4x + 5) \end{aligned}$$

Factors of -40	Sum
1 and -40	-39
2 and -20	-18
4 and -10	-6
5 and -8	-3

Problem 10.5

Factor. Check your answer by multiplying.

$$25x^2 - 15x + 2$$

Problem 10.6

Factor. Check your answer by multiplying.

$$6y^2 - 7y - 10$$

Problem 10.7

Factor. Check your answer by multiplying.

$$a^2 + 13a + 36$$

Problem 10.8

Factor. Check your answer by multiplying.

$$12x^2 + x - 1$$

Multi-Step Factoring

In the following problems the directions will change to *factor completely*. Often a polynomial can be factored using more than one method of factoring. When factoring, we should get in the habit of always looking for a GCF first.

Problem 10.7

Factor *completely*.

$$2x^2y + 10xy + 12y$$

Solution

We will first start by removing the GCF of $2y$. Then we will factor the trinomial as we did in the previous examples.

$$2x^2y + 10xy + 12y$$

$$2y(x^2 + 5x + 6)$$

$$2y(x + 2)(x + 3)$$



*Be sure to include the
GCF in all steps*

Original polynomial

Remove GCF of $2y$

Look for factors of 6 that have a sum of 5

(Since the trinomial has a lead coefficient of 1, we can skip the grouping process and move to the final factorization.)

Problem 10.8

Factor *completely*.

$$3x^2 + 6x - 72$$

Problem 10.9

Factor *completely*.

$$6a^2b - 50ab - 36b$$

Problem 10.10Factor *completely*.

$$2x^3 + 11x^2 + 5x$$

Problem 10.11Factor *completely*.

$$20ax - 36bx - 35ay + 63by$$

Problem 10.12Factor *completely*.

$$5x^2 - 14x - 3$$

Problem 10.13Factor *completely*.

$$5xy^2 - 10xy + 5x$$

Section 11: Factoring Special Forms

Factoring the Difference of Two Squares

In Problem 9.5, we learned how to multiply the sum and difference of two terms. We saw that the middle terms cancelled out and we were left with a difference of two squares. If we reverse the two sides of the equation we can obtain the factors of the difference of two squares.

Factoring the Difference of Two Squares

If A and B are real numbers, variables or algebraic expressions, then

$$A^2 - B^2 = (A + B)(A - B)$$

Therefore, the difference of the squares of two terms can be factored into the product of the sum and difference of those two terms.

NOTE: The SUM of two squares can NOT be factored!

One of the signs of a difference of two squares is that there are only two terms with a minus sign between them -- there is no middle term. You should carefully examine each of the terms to see if they are perfect squares.

First, examine any numerical values to see if they are perfect squares. Below is a list of some common perfect squares:

$1^2 = 1$	$5^2 = 25$	$9^2 = 81$
$2^2 = 4$	$6^2 = 36$	$10^2 = 100$
$3^2 = 9$	$7^2 = 49$	$11^2 = 121$
$4^2 = 16$	$8^2 = 64$	$12^2 = 144$

Remember, if you are unsure about whether a number is a perfect square, you can use your calculator to check!

Next, examine the variables to see if they are perfect squares. Naturally, a term such as x^2 or y^2 are perfect squares, but if we use our previous knowledge about exponents, we can see that *any* even exponent is a perfect square. For example:

$$\begin{aligned}x^2 &= (x)^2 \\x^4 &= (x^2)^2 \\x^6 &= (x^3)^2 \\x^{10} &= (x^5)^2 \\x^{26} &= (x^{13})^2\end{aligned}$$

Problem 11.1

Factor.

$$x^2 - 16$$

Solution

$$\begin{aligned}
 & x^2 - 16 \\
 & \begin{array}{c} \boxed{A^2} - \boxed{B^2} \\ \swarrow \quad \searrow \\ = (x)^2 - (4)^2 \end{array} \\
 & \boxed{(A + B)(A - B)} \\
 & = (x + 4)(x - 4)
 \end{aligned}$$

The binomial involves subtraction and both x^2 and 16 are perfect squares. This is a difference of two squares.

Express each term as a square.

Factor accordingly.

$x^2 - 16$ is one of the common differences of two squares that you may see moving forward in this course. It is helpful to become familiar with the common differences of two squares and their factorizations, such as:

$$\begin{aligned}
 x^2 - 1 &= (x + 1)(x - 1) \\
 x^2 - 4 &= (x + 2)(x - 2) \\
 x^2 - 9 &= (x + 3)(x - 3) \\
 x^2 - 16 &= (x + 4)(x - 4) \\
 x^2 - 25 &= (x + 5)(x - 5) \\
 x^2 - 36 &= (x + 6)(x - 6) \\
 x^2 - 49 &= (x + 7)(x - 7) \\
 x^2 - 64 &= (x + 8)(x - 8) \\
 x^2 - 81 &= (x + 9)(x - 9) \\
 x^2 - 100 &= (x + 10)(x - 10) \\
 &\text{etc.}
 \end{aligned}$$

Problem 11.2

Factor.

$$9x^2 - 25y^2$$

Solution

$$\begin{aligned}
 & 9x^2 - 25y^2 \\
 & \begin{array}{c} \boxed{A^2} - \boxed{B^2} \\ \swarrow \quad \searrow \\ = (3x)^2 - (5y)^2 \end{array} \\
 & \boxed{(A + B)(A - B)} \\
 & = (3x + 5y)(3x - 5y)
 \end{aligned}$$

The binomial involves subtraction and both $9x^2$ and $25y^2$ are perfect squares. This is a difference of two squares.

Express each term as a square.

Factor accordingly.

Problem 11.3

Factor.

$$4x^6 - 49y^4$$

Solution

$$\begin{array}{c}
 4x^6 - 49y^4 \\
 \downarrow \quad \downarrow \\
 = (2x^3)^2 - (7y^2)^2
 \end{array}$$

The binomial involves subtraction and both $4x^6$ and $49y^4$ are perfect squares.

Express each term as a square.

$$= (2x^3 + 7y^2)(2x^3 - 7y^2)$$

Factor accordingly.

Problem 11.4

Factor.

$$81x^2 - 100$$

Problem 11.5

Factor.

$$25a^2 - 36b^2$$

Problem 11.6

Factor.

$$64x^8 - 9y^6$$

Problem 11.7

Factor.

$$49x^2 + 16$$

Multi-Step Factoring

Whenever we are asked to factor, the first thing that we must look for is a GCF. If there is a GCF, factor it out and then factor the resulting polynomial. Also, there are times after you factor a polynomial that there is another step to completely finish factoring. Often times, when there is more than one step to the factorization process, you will be asked to factor *completely*.

Problem 11.8

Factor *completely*.

$$50x^2 - 18y^2$$

Solution

$$\begin{aligned} &50x^2 - 18y^2 \\ &= 2(25x^2 - 9y^2) \\ &\quad \downarrow \quad \downarrow \\ &= 2((5x)^2 - (3y)^2) \\ &= 2(5x + 3y)(5x - 3y) \end{aligned}$$

Be sure to include the GCF in all steps

Original polynomial

Remove the GCF of 2

$25x^2 - 9y^2$ is a difference of two squares.
Express each term as a square.

Factor accordingly.

Problem 11.9

Factor *completely*.

$$16x^4 - 81$$

Solution

$$\begin{aligned} &16x^4 - 81 \\ &= (4x^2)^2 - (9)^2 \\ &= (4x^2 + 9)(4x^2 - 9) \\ &= (4x^2 + 9)((2x)^2 - (3)^2) \\ &= (4x^2 + 9)(2x + 3)(2x - 3) \end{aligned}$$

This is a difference of two squares.

Express each term as a square.

Factor accordingly. Notice that $4x^2 - 9$ is also a difference of two squares!

Factor accordingly.

The factorization is now complete.

At other times, after you factor a polynomial a difference of two squares is revealed, such as in the following:

Problem 11.10

Factor *completely*.

$$x^3 + 5x^2 - 9x - 45$$

Solution

$$x^3 + 5x^2 - 9x - 45$$

Since there are four terms, use factoring by grouping.

$$= x^2(x + 5) - 9(x + 5)$$

Note the common binomial factor.

$$= (x + 5)(x^2 - 9)$$

Note that $(x^2 - 9)$ is a difference of two squares!

$$= (x + 5)((x)^2 - (3)^2)$$

Express each term as a square.

$$= (x + 5)(x + 3)(x - 3)$$

Factor accordingly.

Problem 11.11

Factor *completely*.

$$27x^2 - 75y^2$$

Problem 11.12

Factor *completely*.

$$81x^4 - 16$$

Problem 11.13

Factor *completely*.

$$x^3 + 6x^2 - 4x - 24$$

Problem 11.14

Factor *completely*.

$$36a^2 - 16b^2$$

Factoring the Sum or Difference of Two Cubes

Two other special forms are the sum and difference of two cubes. In each case there are only two terms -- this time perfect cubes -- joined by either an addition sign or a minus sign. The following formulas should be memorized.

Factoring the Sum of Two Cubes

$$A^3 + B^3 = (A + B)(A^2 - AB + B^2)$$

Factoring the Difference of Two Cubes

$$A^3 - B^3 = (A - B)(A^2 + AB + B^2)$$

Notice the similarities and differences between the two formulas.

Before we proceed with solving any problems, it would be helpful to become familiar with some common perfect cubes:

$1^3 = 1$	$3^3 = 27$	$5^3 = 125$	$7^3 = 343$	$9^3 = 729$
$2^3 = 8$	$4^3 = 64$	$6^3 = 216$	$8^3 = 512$	$10^3 = 1000$

Problem 11.15

Factor.

$$8x^3 - 27y^3$$

Solution

$$8x^3 - 27y^3$$

$$\boxed{A^3} - \boxed{B^3}$$

$$(2x)^3 - (3y)^3$$

$$= (A - B)(A^2 + AB + B^2)$$

$$(2x - 3y)((2x)^2 + (2x)(3y) + (3y)^2)$$

$$(2x - 3y)(4x^2 + 6xy + 9y^2)$$

$8x^3$ and $27y^3$ are both perfect cubes. Since the expression involves subtraction, this is a difference of two cubes.

Express each term as a cube. In this case: $A = 2x$ and $B = 3y$.

Substitute according to the formula.

Simplify all terms.

There could also be multi-step problems for the sum or difference of cubes. Again, you are often asked to factor *completely*.

Problem 11.15

Factor *completely*.

$$24x^4 + 3xy^3$$

Solution

$$24x^4 + 3xy^3 = 3x(8x^3 + y^3)$$

Factor out the GCF of $3x$.

$$\begin{array}{c} \boxed{A^3} + \boxed{B^3} \\ \swarrow \quad \searrow \\ 3x((2x)^3 + (y)^3) \end{array}$$

$8x^3 + y^3$ is a sum of two cubes.

Express each term as a cube. In this case:

$A = 2x$ and $B = y$.

$$\boxed{(A + B)(A^2 - AB + B^2)}$$

Substitute according to the formula.

$$3x(2x + y)((2x)^2 - (2x)(y) + (y)^2)$$

Simplify all terms.

Remember to include the GCF in each step and in the final factorization.

$$3x(2x + y)(4x^2 - 2xy + y^2)$$

Problem 11.16

Factor.

$$8x^3 - 27$$

Problem 11.17

Factor.

$$64x^3 + 1$$

Problem 11.18

Factor.

$$64x^3 - 125y^3$$

Problem 11.19

Factor.

$$27x^3 + 8y^3$$

Problem 11.20Factor *completely*.

$$32x^3y - 4y^4$$

Problem 11.21Factor *completely*.

$$54x^4y + 16xy^4$$

Section 12: Solving Polynomial Equations by Factoring

Now that we have discussed all the methods of factoring, we will apply our learning to solving polynomial equations. When solving polynomial equations, we use the following property.

Zero Product Property

If the product of two numbers is 0, then one of the numbers must be equal to 0.

$$\begin{aligned} \text{If } a \cdot b &= 0, \\ \text{then } a &= 0 \text{ or } b = 0 \end{aligned}$$

Problem 12.1

Find all solutions to the equation.

$$3x^2 + 13x + 12 = 0$$

Solution

We begin by factoring the polynomial. We look for two integers whose product is $(3)(12) = 36$, and whose sum is 13. We see that the two numbers needed to split the middle term are 4 and 9.

$$\begin{aligned} 3x^2 + 13x + 12 &= 0 \\ 3x^2 + 4x + 9x + 12 &= 0 \\ x(3x + 4) + 3(3x + 4) &= 0 \\ (3x + 4)(x + 3) &= 0 \end{aligned}$$

We now make use of the Zero Product Property. One of the factors must be zero. We proceed by breaking our equation into two simpler equations, which allows us to solve for x .

$$\begin{array}{ccc} (3x + 4)(x + 3) = 0 & & \\ \swarrow \quad \searrow & & \\ \begin{array}{l} 3x + 4 = 0 \\ 3x = -4 \\ x = -\frac{4}{3} \end{array} & & \begin{array}{l} x + 3 = 0 \\ x = -3 \end{array} \end{array}$$

This equations has two solutions and can be expressed as a solution set.

$$x = \left\{ -\frac{4}{3}, -3 \right\}$$

In order to use the Zero Product Property, our product must first be equal to zero. Often, we must rearrange the equation before factoring and using the Zero Product Property.

Problem 12.2

Find all solutions to the equation.

$$5x^2 = 80$$

Solution

We begin by making one side of the equation zero, and then factor using the appropriate method.

$$\begin{array}{l}
 5x^2 = 80 \\
 5x^2 - 80 = 0 \\
 5(x^2 - 16) = 0 \\
 5(x - 4)(x + 4) = 0 \\
 \begin{array}{ccc}
 \swarrow & \downarrow & \searrow \\
 5 = 0 & x - 4 = 0 & x + 4 = 0 \\
 5 \neq 0 & x = 4 & x = -4
 \end{array}
 \end{array}$$

A constant factor
is *never* zero!

Original equation

Subtract 80 from both sides

Factor out the GCF of 5

Factor the difference of squares

Use the Zero Product Property and set each factor equal to zero

Solve each equation for x

Therefore, the solutions are $x = 4$ or $x = -4$.

Problem 12.3

Find all solutions to the equation.

$$4x^2 - 4x - 3 = 0$$

Problem 12.4

Find all solutions to the equation.

$$5x^2 - 7x = 0$$

Problem 12.5

Find all solutions to the equation:

$$3x^2 = 300$$

Problem 12.6

Find all solutions to the equation:

$$4x^2 = 12x - 5$$

Problem 12.7

Find all solutions to the equation:

$$7x^3 = 28x$$

Problem 12.8

Find all solutions to the equation:

$$x^3 - x^2 - 9x + 9 = 0$$

NOTE: A polynomial equation with degree 2 (the highest exponent on the variable) will have 2 solutions. A polynomial equation with degree 3 will have 3 solutions, and so on...

Translating words into equations

Sometimes, we are faced with a problem that describes a situation and we need to translate the problem into an equation and solve. Let us first write down all the words we can think of that could represent the symbol to the left.

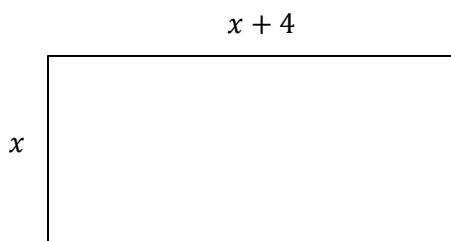
Symbol	Words
$+$	
$-$	
\times	
\div	
$=$	
x Variable	

Problem 12.9

The length of a rectangular garden is 4 meters more than its width. The area of the garden is 96 square meters. Find the dimensions of the garden by using a quadratic equation.

Solution

Whenever we are faced with a problem involving a shape, it is helpful to draw a picture.



Since the length is 4 meters *more than* the width, notice that the width (the shorter side) is labeled x and the length (the longer side) is labeled $x + 4$ in the picture.

We are also given that the area of the rectangle is 96 square meters. Recall that the area of a rectangle is *length \times width*. Therefore, we will multiply the length and width and set it equal to the area.

$$x(x + 4) = 96$$

$$x^2 + 4x = 96$$

$$x^2 + 4x - \mathbf{96} = 96 - \mathbf{96}$$

$$x^2 + 4x - 96 = 0$$

$$(x + 12)(x - 8) = 0$$

$$x + 12 = 0, \quad x - 8 = 0$$

$$x = -12, \quad x = 8$$

Since length cannot be negative, $x = -12$ is not a solution. We are left with only one solution, $x = 8$. Therefore, $x + 4 = 12$.

In conclusion, the width of the rectangle is 8 meters and length of the rectangle is 12 meters.

Problem 12.10

Each side of a square is increased by 4 inches. The area of the new square is 121 square inches. Find the length of a side of the original square.

Problem 12.11

A rectangular garden whose length is 7 feet longer than its width has an area of 60 square feet. Find the dimensions of the garden.

Problem 12.12

If 12 is subtracted from 5 times a number, the result is 6 less than the square of the number. Find the number(s).

Problem 12.13

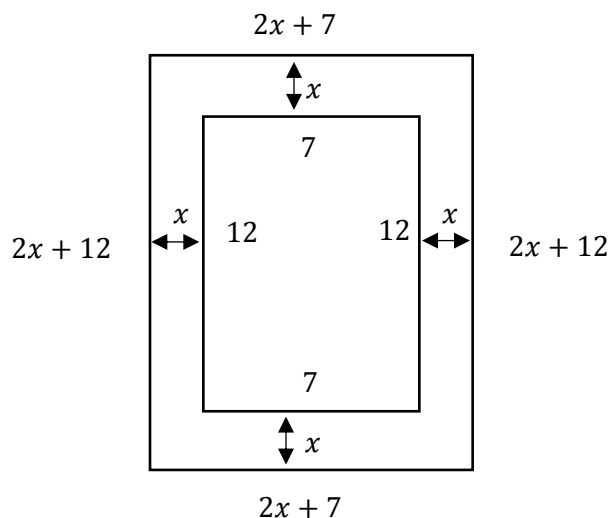
The height of a triangle is 3 yards less than twice the base. The area of the triangle is 27 square yards. Find the base and height of the triangle.

Problem 12.14

A painting measuring 7 inches by 12 inches is surrounded by a frame of uniform width. If the combined area of the painting and the frame is 176 square inches, determine the width of the frame.

Solution

Once again, we will draw a picture to help guide us.



We let the width of the frame be x inches. The width and length of the painting are 7 and 12 inches respectively. So, the outer length becomes $2x + 12$ and the outer width becomes $2x + 7$ inches. Observe that the outer length times the outer width is equal to the combined area. Since we are given that the combined area is 176 square inches, our equation is

$(2x + 12)(2x + 7) = 176$ $4x^2 + 38x + 84 = 176$ $4x^2 + 38x - 92 = 0$ $2x^2 + 19x - 46 = 0$ $2x^2 + 23x - 4x - 46 = 0$ $x(2x + 23) - 2(2x + 23) = 0$ $(2x + 23)(x - 2) = 0$ $2x + 23 = 0, \quad x - 2 = 0$ $x = -\frac{23}{2}, \quad x = 2$ $x = 2$	<p>(Outer L) · (Outer W) = Combined Area</p> <p>Expand</p> <p>Subtract 176 to make one side 0</p> <p>Divide each term by 2</p> <p>Split the middle term</p> <p>Factor by grouping</p> <p>Split into two possibilities</p> <p>Solve</p> <p>Width cannot be negative.</p>
---	---

Therefore, the width of the frame is **2 inches**.

Problem 12.15

A painting measuring 8 inches by 14 inches is surrounded by a frame of uniform width. If the combined area of the painting and the frame is 280 square inches, determine the width of the frame.

Problem 12.16

A painting measuring 9 inches by 13 inches is surrounded by a frame of uniform width. If the combined area of the painting and the frame is 221 square inches, determine the width of the frame.

Section 13: Rational Functions (Simplifying, Multiplying and Dividing)

In this section we will begin our discussion of rational expressions. A **rational function** is a function consisting of a polynomial expression divided by another (non-zero) polynomial expression. When we hear the word “rational”, we should think of fractions.

Examples of rational functions:

$$f(x) = \frac{3x}{4x-1} \quad g(x) = \frac{2x^2}{x^2-x-6} \quad h(x) = \frac{x^2+5}{x^2-16}$$

Domain of a Rational Function

Since we cannot divide by zero, we must exclude from the domain any values that cause the denominator to be zero. Our result should be expressed in interval notation as we did in the inequality section.

Problem 13.1

Find the domain. Express using interval notation.

$$f(x) = \frac{3}{x^2 - 1}$$

Solution

To find the domain, we have to find the value(s) of x that will cause the denominator to be zero. To do this, we simply set the denominator equal to zero. In many cases, the denominator will have to be factored. (The numerator will NOT affect the domain!)

$$x^2 - 1 = 0$$

Set denominator equal to zero

$$(x + 1)(x - 1) = 0$$

Factor (difference of squares)

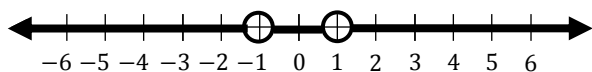
$$x + 1 = 0 \quad \text{or} \quad x - 1 = 0$$

Set each factor equal to zero

$$x = -1 \quad \text{or} \quad x = 1$$

Solve the equations

The values -1 and 1 will make the denominator zero, so these are the values to exclude. A number line can assist in expressing the domain using interval notation.



Domain of $f = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

The hollow circles show that -1 and 1 are not in the domain while the shaded portions of the number line shows the values that are in the domain.

The **domain of a rational function** is the set of all real numbers **EXCEPT** the numbers that make the denominator zero.

Problem 13.2

Find the domain. Express using interval notation.

$$f(x) = \frac{3x}{4x - 1}$$

Problem 13.3

Find the domain. Express using interval notation.

$$g(x) = \frac{2x^2}{x^2 - x - 6}$$

Problem 13.4

Find the domain. Express using interval notation.

$$h(x) = \frac{x^2 + 5}{x^2 - 16}$$

Problem 13.5

Find the domain. Express using interval notation.

$$f(x) = \frac{2}{x^2 - 3x}$$

Simplifying Rational Expressions

A rational expression is in **simplest form** if the numerator and denominator have no common factors. To simplify a rational expression, we must first factor both the numerator and the denominator then cancel any factors that appear in BOTH the numerator and denominator.

Problem 13.6

Simplify.

$$\frac{x^2 - 4x + 3}{x - 1}$$

Solution

Factor the numerator of the expression. The denominator is already in simplest form.

$\frac{x^2 - 4x + 3}{x - 1}$	This is the original expression
$= \frac{(x - 3)(x - 1)}{x - 1}$	Factor the trinomial in the numerator
$= \frac{(x - 3)\cancel{(x - 1)}}{\cancel{x - 1}}$	Cancel the common factor $(x - 1)$
$= x - 3$	This is the simplified form

There are no factors remaining in the denominator so $(x - 3)$ is the simplified expression.

Note: If the only remaining factor is in the denominator, a numerator of 1 would be necessary.

Be careful with binomials that contain subtraction! If a rational expression contains factors that are opposites, they will cancel and a SINGLE factor of -1 will remain.

Example: Simplify $\frac{x^2 - 25}{15 - 3x}$

$$\frac{x^2 - 25}{15 - 3x} = \frac{(x + 5)\cancel{(x - 5)}}{3\cancel{(5 - x)}} = -\frac{x + 5}{3}$$

NOTE: When simplifying rational expressions, you can only cancel **factors** that are common to the numerator and denominator. It is incorrect to cancel **terms** from the numerator and denominator.

INCORRECT!!! $\frac{\cancel{x-7}}{\cancel{x}} \neq -7$	or	$\frac{\cancel{x^2+2x-15}}{\cancel{x^2+3}} \neq \frac{2x-15}{3}$ INCORRECT!!!
--	----	---

Multiplying Rational Expressions

When multiplying two (or more) rational expressions, we generally multiply the numerators and multiply the denominators. However, before multiplying, we will first factor each numerator and denominator. Doing so will allow us to first simplify by canceling any factors common to both a numerator AND a denominator. Then we will multiply.

Problem 13.7

Simplify.

$$\frac{x^2 - 9}{x - 4} \cdot \frac{x^2 - 2x - 8}{2x - 6}$$

Solution

Instead of multiplying the numerators and denominators, we will first factor and simplify to make the process simpler.

$\frac{x^2 - 9}{x - 4} \cdot \frac{x^2 - 2x - 8}{2x - 6}$	Original expression
$= \frac{(x - 3)(x + 3)}{x - 4} \cdot \frac{(x - 4)(x + 2)}{2(x - 3)}$	Factor each numerator and denominator
$= \frac{\cancel{(x - 3)}(x + 3)}{\cancel{x - 4}} \cdot \frac{\cancel{(x - 4)}(x + 2)}{2\cancel{(x - 3)}}$	Cancel factors common to both a numerator AND a denominator
$= \frac{(x + 3)(x + 2)}{2}$	Multiply any remaining factors in numerator and/or denominator

The final expression should be kept in factored form. It is unnecessary to multiply the binomials in the numerator.

Dividing Rational Expressions

Now that we know how to multiply rational expressions, dividing should come easily! Let's recall how to divide two rational numbers.

To divide rational numbers, **multiply by the reciprocal**.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$

Problem 13.8

Simplify.

$$\frac{5x^2 - 15x}{x^2 + 4x - 12} \div \frac{x - 3}{x^2 - 4}$$

Solution

In order to divide two rational expressions, we keep the first expressions the same, change the operation to multiplication, and take the reciprocal of the second expression. Then follow the steps for multiplying rational expressions.

$$\frac{5x^2 - 15x}{x^2 + 4x - 12} \div \frac{x - 3}{x^2 - 4}$$

Original expression

$$= \frac{5x^2 - 15x}{x^2 + 4x - 12} \cdot \frac{x^2 - 4}{x - 3}$$

Multiply by the reciprocal

$$= \frac{5x(x - 3)}{(x + 6)(x - 2)} \cdot \frac{(x + 2)(x - 2)}{x - 3}$$

Factor ALL numerators and denominators

$$= \frac{5x(\cancel{x - 3})}{(x + 6)(\cancel{x - 2})} \cdot \frac{(x + 2)(\cancel{x - 2})}{\cancel{x - 3}}$$

Cancel factors that are common to BOTH the numerator and denominator

$$= \frac{5x(x + 2)}{x + 6}$$

Simplify

Problem 13.9

Simplify.

$$\frac{2x + 1}{2x^2 - x - 1}$$

Problem 13.10

Simplify.

$$\frac{20 - 5y}{y^2 - 16}$$

Problem 13.11

Simplify.

$$\frac{x^2 - 25}{x} \cdot \frac{x + 2}{x^2 - 3x - 10}$$

Problem 13.12

Simplify.

$$\frac{x^2 - 5x + 6}{1 - x} \cdot \frac{x^2 - 1}{4x - 8}$$

Problem 13.13

Simplify.

$$\frac{x^2 + 5x + 4}{2x} \div \frac{x^2 + x - 12}{x^2 - 3x}$$

Problem 13.14

Simplify.

$$\frac{y^2 + y}{y^2 - 4} \div \frac{y^2 - 1}{y^2 + 5y + 6}$$

Adding and Subtracting Rational Expressions

When it comes to adding and subtracting rational expression (or fractions), we must first have common denominators. In such cases, we add/subtract the numerators and the denominator remains the same.

Problem 13.15

Add.

$$\frac{2x + 1}{x - 5} + \frac{x^2 - 5x + 2}{x - 5}$$

Solution

Since the denominators are the same, $(x - 5)$, we add the numerators and keep the common denominator.

$\frac{2x + 1}{x - 5} + \frac{x^2 - 5x + 2}{x - 5}$	Original expression
$= \frac{2x + 1 + x^2 - 5x + 2}{x - 5}$	Add the numerators and place over the common denominator
$= \frac{x^2 - 3x + 3}{x - 5}$	Simplify the numerator

NOTE: If possible, we should factor and simplify the resulting expression.

Problem 13.16

Subtract.

$$\frac{y^2}{y^2 - 25} - \frac{10y - 25}{y^2 - 25}$$

Solution

Once again the denominators are the same so we will subtract the expressions. When subtracting rational expressions, it is important to include parentheses around the second numerator so we are sure to subtract the ENTIRE polynomial.

$\frac{y^2}{y^2 - 25} - \frac{10y - 25}{y^2 - 25}$	Original expression
$= \frac{y^2 - (10y - 25)}{y^2 - 25}$	Subtract numerators being sure to distribute the negative to the second polynomial.
$= \frac{y^2 - 10y + 25}{y^2 - 25}$	Simplify
$= \frac{(y - 5)(y - 5)}{(y + 5)(y - 5)}$	Factor both the numerator and denominator then simplify
$= \frac{y - 5}{y + 5}$	

When the denominators are not the same, we must find the Least Common Denominator (LCD) of the expressions and have each expression have the same denominators.

Adding/Subtracting Ration Expressions with Different Denominators

Find the LCD of the expressions.

Rewrite expressions as equivalent rational expressions with common denominators.

(multiply numerator AND denominator by the factor needed to create the LCD in each expression)

Add/subtract numerators (using parentheses when subtracting) and place over the LCD.

If possible, factor and simplify the rational expression

Problem 13.17

Subtract.

$$\frac{4}{10x} - \frac{3}{5x^2}$$

Solution

In order to subtract, we must first find the LCD of the expressions. To find the LCD, it may be helpful to list all the factors of each denominator.

$$10x = 5 \cdot 2 \cdot x$$

$$5x^2 = 5 \cdot x \cdot x$$

The LCD will be all the factors of the first denominator ($5, 2, x$) and any unlisted factors from the second denominator (an additional factor of x)

$$\text{LCD} = 5 \cdot 2 \cdot x \cdot x = 10x^2$$

Once we recognize the LCD, we must now write equivalent expressions that have the same denominators ($10x^2$)

Multiply numerator and denominator by x to obtain $10x^2$, the LCD

$$\frac{4}{10x} \cdot \frac{x}{x} - \frac{3}{5x^2} \cdot \frac{2}{2} = \frac{4x}{10x^2} - \frac{6}{10x^2}$$

Multiply numerator and denominator by 2 to obtain $10x^2$, the LCD

Next, subtract the numerators and place over the common denominator. Our final step, when necessary, will be to factor and simplify the expression.

$$\frac{4x-6}{10x^2} = \frac{\cancel{2}(x-3)}{\cancel{2}10x^2} = \frac{x-3}{5x^2}$$

Divide by a factor of 2

Now let's take a look at an example that contains more complicated denominators that must be factored.

Problem 13.18

Add.

$$\frac{x+2}{x^2-9} + \frac{3x}{x+3}$$

Solution

In this example the denominators are binomials. We can start by listing the factors of each denominator.

$$\left. \begin{array}{l} x^2 - 9 = (x - 3)(x + 3) \\ x + 3 = (x + 3) \end{array} \right\} \text{LCD} = (x - 3)(x + 3)$$

Since the first expression already contains the LCD, we only have to find an equivalent expression for the second expression.

$\frac{x+2}{x^2-9} + \frac{3x}{x+3}$	Original expression
$= \frac{x+2}{(x-3)(x+3)} + \frac{3x}{x+3} \cdot \frac{(x-3)}{(x-3)}$	Multiply the numerator and denominator of the second expression by the missing factor $(x - 3)$
$= \frac{x+2+3x(x-3)}{(x-3)(x+3)}$	Add the numerator and place over the common denominator
$= \frac{x+2+3x^2-9x}{(x-3)(x+3)}$	Distribute in the numerator
$= \frac{3x^2-8x+2}{(x-3)(x+3)}$	Simplify. (if possibly factor the numerator and simplify further)

Problem 13.19

Subtract.

$$\frac{x^2+6x+2}{x^2+x-6} - \frac{2x-1}{x^2+x-6}$$

Problem 13.20

Add.

$$\frac{7}{3x^2} + \frac{5}{2x}$$

Problem 13.21

Add.

$$\frac{5x}{x-4} + \frac{8}{x^2-16}$$

Problem 13.22

Subtract.

$$\frac{x-2}{x^2-5x} - \frac{4x}{x-5}$$

Problem 13.23

Add.

$$\frac{2y+9}{y^2-7y+12} + \frac{2}{y-3}$$

Problem 13.24

Subtract.

$$\frac{x-1}{x-3} - \frac{x}{x+3}$$

Section 14: Complex Rational Expressions

A **complex rational expression** is a rational expression that has one or more rational expressions in its numerator and/or denominator. For example, the following is a complex rational expression:

$$\frac{\frac{1}{x} + \frac{1}{x^2}}{1 + \frac{1}{x}}$$

Simplifying Complex Rational Expressions

There are several ways to simplify a complex rational expression. One method is as follows:

Find the least common denominator (LCD) of all the rational expressions in the numerator and denominator.

Make the denominators of all the rational expressions into the LCD by whatever means is necessary. *(Remember, you must treat all numerators the same as the denominators!)*

Cross off the denominators of the rational expressions in both the numerator and denominator, leaving only the numerators.

Simplify the resulting rational expression, if possible.

Problem 14.1

Simplify.

$$\frac{\frac{1}{x} + \frac{1}{x^2}}{1 + \frac{1}{x}}$$

Solution

$$\frac{\frac{1}{x} + \frac{1}{x^2}}{1 + \frac{1}{x}}$$

$$\frac{\frac{1(x)}{x(x)} + \frac{1}{x^2}}{\frac{1(x^2)}{1(x^2)} + \frac{1(x)}{x(x)}} = \frac{\frac{x}{x^2} + \frac{1}{x^2}}{\frac{x^2}{x^2} + \frac{x}{x^2}}$$

$$\frac{\frac{x}{\cancel{x^2}} + \frac{1}{\cancel{x^2}}}{\frac{x^2}{\cancel{x^2}} + \frac{x}{\cancel{x^2}}} = \frac{x + 1}{x^2 + x}$$

$$\frac{x + 1}{x^2 + x} = \frac{\cancel{x} + 1}{x(\cancel{x} + 1)} = \frac{1}{x}$$

The first step is to find the LCD of all the denominators. Since 1 did not have a denominator in the original problem, we should list it over 1. Examining the denominators of all the rational expressions, we find that x^2 is the LCD.

Change all denominators to the LCD, x^2 , by multiplying the denominators and the numerators of the rational expressions by whatever is necessary to accomplish this.

Cross off the denominators of all the rational expressions, leaving only the numerators.

Simplify the remaining rational expression, if possible.

This method works because when we reach the point when all of the rational expressions are over the same denominator (the LCD), we could choose to multiply the top and bottom of the expression by the LCD (*anything over itself equals 1 so we are not changing the value of the expression, just its form*).

When we do that, all of the denominators would cancel out as follows:

$$\frac{\frac{x}{x^2} + \frac{1}{x^2}}{\frac{x^2}{x^2} + \frac{x}{x^2}} \cdot \frac{x^2}{x^2} = \frac{x+1}{x^2+x}$$

Now you would continue to simplify the remaining rational expression, if possible.

Another method is to **simplify a complex rational expression by dividing** as follows:

Add or subtract the terms in the numerator to get a single rational expression.

Add or subtract the terms in the denominator to get a single rational expression.

Perform division of the numerator by the denominator by inverting the denominator and then multiplying.

Simplify the resulting rational expression, if possible.

Problem 14.2

Simplify.

$$\frac{\frac{x}{3} - \frac{3}{x}}{\frac{1}{3} + \frac{1}{x}}$$

Solution

$$\begin{aligned} & \text{The LCD is } 3x \\ & \frac{\frac{x}{3} - \frac{3}{x}}{\frac{1}{3} + \frac{1}{x}} = \frac{\frac{x(x)}{3(x)} - \frac{(3)3}{(3)x}}{\frac{1(x)}{3(x)} + \frac{(3)1}{(3)x}} \\ & \text{The LCD is } 3x \\ & = \frac{\frac{x^2}{3x} - \frac{9}{3x}}{\frac{x}{3x} + \frac{3}{3x}} = \frac{\frac{x^2 - 9}{3x}}{\frac{x + 3}{3x}} = \frac{x^2 - 9}{3x} \cdot \frac{3x}{x + 3} \\ & = \frac{x^2 - 9}{\cancel{3x}} \cdot \frac{\cancel{3x}}{x + 3} = \frac{x^2 - 9}{x + 3} \\ & \frac{(x + 3)(x - 3)}{x + 3} = \frac{\cancel{(x + 3)}(x - 3)}{\cancel{(x + 3)}} = x - 3 \end{aligned}$$

Add the rational expressions in the numerator and the denominator to get a single rational expression in each.

Invert the denominator and multiply.

Simplify the remaining rational expression, if possible.

NOTE: Either method can be used to simplify a complex rational expression.

Problem 14.3

Simplify.

$$\frac{\frac{x}{y} - 1}{\frac{x^2}{y^2} - 1}$$

Problem 14.4

Simplify.

$$\frac{\frac{1}{5} + \frac{1}{x}}{\frac{x}{5} - \frac{5}{x}}$$

Problem 14.5

Simplify.

$$\frac{\frac{1}{x} + \frac{y}{x^2}}{\frac{1}{y} + \frac{x}{y^2}}$$

Problem 14.6

Simplify.

$$\frac{\frac{7}{x+7}}{\frac{1}{x+7} - \frac{1}{x}}$$

Section 15: Rational Equations

In previous sections we learned how to add, subtract, multiply, divide and simplify rational expressions. A **rational equation** is an equation that contains one or more rational expressions.

Solving Rational Equations

The simplest type of rational equation is one where you have one rational expression set equal to another. You can solve this type of equation by following these steps:

Determine the domain restrictions. List whatever values make the denominator zero.

Cross-multiply.

Solve the resulting equation.

Reject any solution that is on your list of restricted domain values. Check any other solution by substituting that value back into the *original equation*.

Note: If your solution does end up being one of the restricted domain values, then we say that the equation has **no solution**.

Problem 15.1

Solve for x and check.

$$\frac{x+2}{x+10} = \frac{x-3}{x+4}$$

Solution

$$\begin{array}{l} x+10=0 \quad \text{and} \quad x+4=0 \\ x=-10 \quad \quad \quad x=-4 \end{array}$$

Find the domain restrictions.

In this case, $x \neq -10, -4$.

$$\frac{x+2}{x+10} \times \frac{x-3}{x+4}$$

Cross-multiply.

$$(x+2)(x+4) = (x+10)(x-3)$$

Solve the resulting equation.

$$x^2 + 6x + 8 = x^2 + 7x - 30$$

$$x^2 - x^2 + 6x + 8 = x^2 - x^2 + 7x - 30$$

$$6x - 6x + 8 = 7x - 6x - 30$$

$$8 + 30 = x - 30 + 30$$

$$38 = x$$

Check: $x = 38$

$$\frac{x+2}{x+10} = \frac{x-3}{x+4}$$

$$\frac{38+2}{38+10} \neq \frac{38-3}{38+4}$$

$$\frac{40}{48} \neq \frac{35}{42}$$

$$\frac{5}{6} = \frac{5}{6} \checkmark$$

Check that the solution is not one of the restricted domain values. Since in this case, $x = 38$ is not a restricted value, we can go ahead and check the solution in the original equation.

However, not all rational equations are in the form of one rational expression is set equal to another. If this is this case, then follow the steps below:

Determine the domain restrictions. List whatever values make the denominator zero.

Find the LCD of all denominators in the equation.

Make all denominators in the equation into the LCD. (*Remember that you must treat the numerators the same as the denominators!*)

Once you are sure that all expressions are over the LCD, cross off all denominators.

Solve the resulting equation.

Reject any solution that is on your list of restricted domain values. Check any other solution by substituting that value back into the *original equation*.

Again, if one your solution ends up being one of the restricted domain values, then we say that the equation has **no solution**.

NOTE: This method works because when all of the rational expressions in the equation are over the same denominator (the LCD), we could choose to multiply both sides of the equation by the LCD. When we do that, all of the denominators would cancel out as in the following example:

$$(\cancel{3x}) \left(\frac{x+1}{\cancel{3x}} + \frac{2}{\cancel{3x}} \right) = \left(\frac{x+5}{\cancel{3x}} \right) (\cancel{3x})$$

We would then go ahead and solve the resulting equation:

$$x + 1 + 2 = x + 5.$$

In other words, crossing off all of the denominators is a “short cut”.

Problem 15.2Solve for x and check.

$$\frac{x+6}{2x} + \frac{x+24}{5x} = 2$$

Solution

$$\begin{array}{l} 2x = 0 \text{ and } 5x = 0 \\ x = 0 \qquad \qquad x = 0 \end{array}$$

Find the domain restrictions.

In this case, $x \neq 0$.

$$\frac{x+6}{2x} + \frac{x+24}{5x} = \frac{2}{1}$$

This is the original equation, with all terms listed with a denominator.

$$\frac{5(x+6)}{5(2x)} + \frac{2(x+24)}{2(5x)} = \frac{2(10x)}{1(10x)}$$

Change all denominators into the LCD, $10x$.

$$\frac{5(x+6)}{10x} + \frac{2(x+24)}{10x} = \frac{2(10x)}{10x}$$

$$\frac{5(x+6)}{10\cancel{x}} + \frac{2(x+24)}{10\cancel{x}} = \frac{2(10x)}{10\cancel{x}}$$

Cross off all denominators.

$$\begin{aligned} 5(x+6) + 2(x+24) &= 2(10x) \\ 5x + 30 + 2x + 48 &= 20x \\ 7x + 78 &= 20x \\ 7x - 7x + 78 &= 20x - 7x \\ 78 &= 13x \\ 6 &= x \end{aligned}$$

Solve the resulting equation.

Check: $x = 6$

$$\frac{x+6}{2x} + \frac{x+24}{5x} = 2$$

$$\frac{6+6}{2(6)} + \frac{6+24}{5(6)} \neq 2$$

$$\frac{12}{12} + \frac{30}{30} \neq 2$$

$$1 + 1 = 2 \checkmark$$

Check that the solution is not one of the restricted domain values. Since in this case, $x = 6$ is not a restricted value, we can go ahead and check the solution in the original equation.

Problem 15.3Solve for x and check.

$$\frac{32}{x^2 - 25} - \frac{4}{x + 5} = \frac{2}{x - 5}$$

Solution

$$\begin{aligned} x + 5 = 0 & \text{ and } x - 5 = 0 \\ x = -5 & \quad x = 5 \end{aligned}$$

Find the domain restrictions. If we factor, $x^2 - 25 = (x + 5)(x - 5)$ we can see that this is the LCD. In this case, $x \neq -5, 5$.

$$\frac{32}{(x + 5)(x - 5)} - \frac{4}{x + 5} = \frac{2}{x - 5}$$

This is the original equation, with all denominators factored.

$$\frac{32}{(x + 5)(x - 5)} - \frac{4(x - 5)}{(x + 5)(x - 5)} = \frac{2(x + 5)}{(x - 5)(x + 5)}$$

Change all denominators into the LCD, $(x + 5)(x - 5)$, supplying whatever is missing.

$$\frac{32}{\cancel{(x + 5)}\cancel{(x - 5)}} - \frac{4(x - 5)}{\cancel{(x + 5)}\cancel{(x - 5)}} = \frac{2(x + 5)}{\cancel{(x - 5)}\cancel{(x + 5)}}$$

Cross off all denominators.

$$\begin{aligned} 32 - 4(x - 5) &= 2(x + 5) \\ 32 - 4x + 20 &= 2x + 10 \\ -4x + 4x + 52 &= 2x + 4x + 10 \\ 52 - 10 &= 6x + 10 - 10 \\ 42 &= 6x \\ 7 &= x \end{aligned}$$

Solve the resulting equation.

Check: $x = 7$

$$\frac{32}{x^2 - 25} - \frac{4}{x + 5} = \frac{2}{x - 5}$$

$$\frac{32}{(7 + 5)(7 - 5)} - \frac{4}{7 + 5} \stackrel{?}{=} \frac{2}{7 - 5}$$

Check that the solution is not one of the restricted domain values. Since in this case, $x = 7$ is not a restricted value, we can go ahead and check the solution in the original equation.

$$\frac{32}{(12)(2)} - \frac{4}{12} \stackrel{?}{=} \frac{2}{2}$$

$$\frac{32}{24} - \frac{4}{12} \stackrel{?}{=} 1$$

$$\frac{32}{24} - \frac{4(2)}{12(2)} \stackrel{?}{=} 1$$

$$\frac{32}{24} - \frac{8}{24} \stackrel{?}{=} 1$$

$$\frac{24}{24} = 1 = 1 \checkmark$$

Problem 15.4

Solve, listing all domain restrictions. Check.

$$\frac{4}{x-3} = \frac{6}{x+3}$$

Problem 15.5

Solve, listing all domain restrictions. Check.

$$\frac{x-2}{x+6} = \frac{x-3}{x+1}$$

Problem 15.6

Solve, listing all domain restrictions. Check.

$$\frac{3x}{x+1} + \frac{4}{x-2} = 3$$

Problem 15.7

Solve, listing all domain restrictions. Check.

$$\frac{x}{x-3} - \frac{3}{x-3} = 9$$

Problem 15.8

Solve, listing all domain restrictions. Check.

$$\frac{7x}{x^2 - 4} + \frac{5}{x - 2} = \frac{2x}{x^2 - 4}$$

Problem 15.9

Solve, listing all domain restrictions. Check.

$$\frac{2x}{x - 3} + \frac{28}{x^2 - 9} = -\frac{6}{x + 3}$$

Section 16: Literal Equations

Sometimes, when equations have more than one variable, it is desirable to solve for or isolate one of the variables and write it in terms of the other variables. For example, this comes up very often when graphing lines. The operations used to solve these equations remain the same as before.

Problem 16.1

Solve for y .

$$2x + 3y + 4 = y + 10$$

Solution

Since we want to solve for (or isolate) y , we will make sure that all the terms involving y are moved to the same side of the equation. We can begin by subtracting y from both sides.

$$\begin{aligned} 2x + 3y - y + 4 &= y - y + 10 \\ 2x + 2y + 4 &= 10 \end{aligned}$$

Now, since the “ y – terms” have been moved to the left hand side, we will make sure to move all the other terms (the terms without y) to the right hand side.

$$\begin{aligned} 2x - 2x + 2y + 4 &= -2x + 10 \\ 2y + 4 &= -2x + 10 \\ 2y + 4 - 4 &= -2x + 10 - 4 \\ 2y &= -2x + 6 \end{aligned}$$

Finally, we can isolate y by dividing each term by 2.

$$\begin{aligned} \frac{2y}{2} &= -\frac{2x}{2} + \frac{6}{2} \\ y &= -x + 3 \end{aligned}$$

Our final answer is $y = -x + 3$.



Problem 16.2Solve for z :

$$x = \frac{2}{3}yz$$

Solution:

Since we need to isolate z , we must remove all the other integers and variables from the right hand side. First, we remove the denominator by multiplying by 3 on both sides.

$$\begin{aligned} (3)(x) &= (3)\frac{2}{3}yz \\ 3x &= 2yz \end{aligned}$$

Now, we must remove $2y$ from the right hand side. Since $2y$ is multiplied by z , we may isolate z by just dividing by $2y$.

$$\begin{aligned} \frac{3x}{2y} &= \frac{2yz}{2y} \\ \frac{3x}{2y} &= z \end{aligned}$$

Therefore, our final answer is $z = \frac{3x}{2y}$.

Problem 16.3Solve for y :

$$2x + 3y + 5 = 8x - 16$$

Problem 16.4Solve for p :

$$pqr = 5$$

Problem 16.5Solve for t :

$$rt + s = 15$$

Problem 16.6Solve for h :

$$g(1 + h) = k$$

Some problems are slightly different in nature. We may be faced with many variables or integers in the denominator. In these cases, we must look for the LCD first.

Problem 16.7

Solve for a :

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$$

Solution:

Notice that the LCD here is abc and that there are three terms. Therefore, we multiply each term by the appropriate expression to make all denominators equal to abc .

$$\frac{1(bc)}{a(bc)} + \frac{1(ac)}{b(ac)} = \frac{1(ab)}{c(ab)}$$

$$\frac{bc}{abc} + \frac{ac}{abc} = \frac{ab}{abc}$$

$$bc + ac = ab$$

$$bc + ac - ac = ab - ac$$

$$bc = ab - ac$$

$$bc = a(b - c)$$

$$\frac{bc}{b - c} = \frac{a(b - c)}{b - c}$$

$$\frac{bc}{b - c} = a$$

Expand.

Cancel the denominators.

Subtract ac from each side to move all the terms involving a to the same side.

Simplify.

Factor out a .

Divide both sides by $b - c$.

Simplify.

Therefore, our final answer is $a = \frac{bc}{b-c}$.

■

Problem 16.8Solve for y :

$$ax + by = cy$$

Problem 16.9Solve for p :

$$\frac{1}{p} - \frac{1}{q} = \frac{1}{r}$$

Problem 16.10Solve for t :

$$A = P(1 + rt)$$

Problem 16.11Solve for h :

$$\frac{2}{h + k} = m$$

Section 17: Radical Expressions and Functions.

Radicals are the reverse operation of applying exponents. A power can be undone with a radical and a radical can be undone with a power.

Generally, if $b^2 = a$, then b is the **square root of a** . We use the symbol $\sqrt{\quad}$ to denote the positive or principal square root of a number. For example:

$$\begin{aligned}\text{If } 2^2 = 4, \text{ then } \sqrt{4} &= \sqrt{2^2} = 2 \\ \text{If } 6^2 = 36, \text{ then } \sqrt{36} &= \sqrt{6^2} = 6\end{aligned}$$

The square root symbol $\sqrt{\quad}$ is called a **radical sign**. The number under the radical sign is called the **radicand**. We refer to the radical sign and its radicand as a **radical expression**.

NOTE: We cannot have a negative number *under* a square root sign. However, we can have a negative sign *outside* of a square root sign to denote the negative square root as follows:

$$-\sqrt{49} = -\sqrt{7^2} = -7$$

You might be most familiar with the square root symbol, but there are many more radical symbols. You can cube numbers, raise them to the fourth power, to the fifth power, etc. Just as with square roots, the reverse operation of cubing a number is to take the *cube root* of a number, the reverse operation of raising a number to the fourth power is to take the fourth root, etc. These other radical signs use a number called an *index* to identify the type of root it represents.

The index is 3 which indicates a cube root.

$$\sqrt[3]{27}$$

In this case, the index is 4, which indicates a fourth root.

$$\sqrt[4]{16}$$

For example:

$$\begin{aligned}\text{If } 3^3 = 27, \text{ then } \sqrt[3]{27} &= \sqrt[3]{3^3} = 3 \\ \text{If } 2^4 = 16, \text{ then } \sqrt[4]{16} &= \sqrt[4]{2^4} = 2\end{aligned}$$

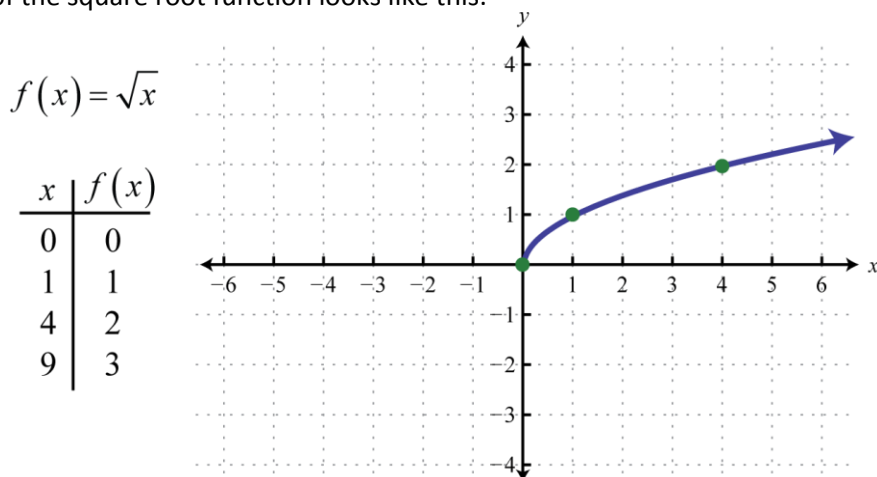
Problem 17.1 Evaluate: $\sqrt{64}$	Problem 17.2 Evaluate: $-\sqrt{81}$	Problem 17.3 Evaluate: $\sqrt{25}$	Problem 17.4 Evaluate: $-\sqrt{100}$
Problem 17.5 Evaluate: $\sqrt[3]{125}$	Problem 17.6 Evaluate: $-\sqrt[4]{81}$	Problem 17.7 Evaluate: $\sqrt[5]{32}$	Problem 17.8 Evaluate: $\sqrt[3]{-64}$

The Square Root Function

The square root function is represented by:

$$f(x) = \sqrt{x}$$

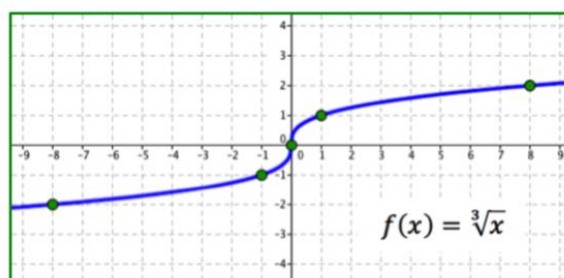
The graph of the square root function looks like this:



Notice that the graph lives in the first quadrant, which means that the domain and range are always non-negative. The domain of this function is $[0, \infty)$.

The Cube Root Function

By contrast, the cube root function, $f(x) = \sqrt[3]{x}$ has a domain and range that is all real numbers, or $(-\infty, \infty)$. A look at the cube root function will support this fact. The same applies to *any odd root*.



Finding the Domain of a Square Root Function

From the discussion above we know that the domain of $f(x) = \sqrt{x}$ is $[0, \infty)$. In general, the domain for any square root function is the set of real numbers for which the radicand is nonnegative, which means *greater than or equal to zero*. The same applies to *any even root*.

Problem 17.9

Find the domain of the following function using interval notation.

$$f(x) = \sqrt{2x + 14}$$

Solution The domain is the set of real number for which $2x + 14$ is non-negative, which means greater than or equal to zero. Therefore, we need to solve the following inequality in order to find the domain:

$$2x + 14 \geq 0$$

$$2x \geq -14$$

$$x \geq -7$$

The domain of $f(x) = \sqrt{2x + 14}$ is $[-7, \infty)$.

Problem 17.10

Find the domain of the following function using interval notation.

$$f(x) = \sqrt{x - 6}$$

Problem 17.11

Find the domain of the following function using interval notation.

$$f(x) = \sqrt{3x - 12}$$

Problem 17.12

Find the domain of the following function using interval notation.

$$f(x) = \sqrt{5x + 30}$$

Problem 17.13

Find the domain of the following function using interval notation.

$$f(x) = \sqrt{10 - 2x}$$

Evaluating Radical Functions

Problem 17.14

Evaluate $f(3)$ for the given square root function $f(x) = \sqrt{4x - 8}$

Solution

$$\begin{aligned} f(x) &= \sqrt{4x - 8} \\ f(\quad) &= \sqrt{4(\quad) - 8} \\ f(3) &= \sqrt{4(3) - 8} \\ f(3) &= \sqrt{12 - 8} \\ f(3) &= \sqrt{4} = 2 \end{aligned}$$

Original equation.

Remove x 's and replace with parentheses.

Inside each parentheses write the value of 3.

Evaluate.

The same concept applies for cube root functions or any other root.

Problem 17.15

Evaluate $f(4)$ for the given square root function

$$f(x) = \sqrt{5x - 4}$$

Problem 17.16

Evaluate $f(-3)$ for the given square root function

$$f(x) = \sqrt{2x + 15}$$

Problem 17.17

Evaluate $f(13)$ for the given cube root function

$$f(x) = \sqrt[3]{2x + 1}$$

Problem 17.18

Evaluate $f(-5)$ for the given cube root function

$$f(x) = \sqrt[3]{2x + 2}$$

Simplifying Radical Expressions

We now know that it is only possible to have non-negative numbers under a square root radical sign (the same holds for any even root) and that positive or negative numbers can be under a cube root sign (again, the same holds for any odd root.) Therefore, the following rules apply:

Simplifying expressions of the form $\sqrt[n]{a^n}$

For any real number a ,

1. If n is even, $\sqrt[n]{a^n} = |a|$
2. If n is odd, $\sqrt[n]{a^n} = a$

Problem 17.19

Simplify.

$$\sqrt{16x^6}$$

Solution

Since this is a square root, $n = 2$, which is even.

$$16x^6 = (4x^3)^2$$

$$\text{Therefore, } \sqrt{16x^6} = \sqrt{(4x^3)^2}$$

$$= |4x^3|$$

$$= 4|x^3|$$

Problem 17.20

Simplify.

$$\sqrt[3]{-27x^3}$$

Solution

Since this is a cube root, $n = 3$, which is odd.

$$-27x^3 = (-3x)^3$$

$$\text{Therefore } \sqrt[3]{(-3x)^3} = -3x.$$

Problem 17.21

Simplify.

$$\sqrt[3]{64x^6}$$

Problem 17.22

Simplify.

$$\sqrt{81x^8}$$

Problem 17.23

Simplify.

$$\sqrt[3]{-8x^{12}}$$

Problem 17.24

Simplify.

$$\sqrt[5]{32x^5}$$

Section 18: Rational Exponents

Rational exponents are exponents that are fractions. *These exponents follow all of the same rules as integral exponents* (Section 8).

The Definition of $a^{\frac{1}{n}}$

If $\sqrt[n]{a}$ is a real number, and $n \geq 2$ and is an integer then,

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

\uparrow \uparrow
 denominator of exponent index of radical sign

Notice how the denominator of the exponent is the index of the radical sign.

RECALL: If n is even, then a must be non-negative. If n is odd, then a can be any real number.

Problem 18.1

Rewrite using radical notation. Simplify, if possible.

$$(25)^{\frac{1}{2}}$$

Solution

$$(25)^{\frac{1}{2}} = \sqrt{25} = \sqrt{5^2} = 5$$

Problem 18.2

Rewrite using radical notation. Simplify, if possible.

$$(-8)^{\frac{1}{3}}$$

Solution

$$(-8)^{\frac{1}{3}} = \sqrt[3]{-8} = \sqrt[3]{(-2)^3} = -2$$

We can also apply the definition in reverse by changing from radical notation to rational exponents. Remember, the index of the radical is the denominator of the rational exponent. Also, it is important to put the entire radicand in a set of parentheses to indicate that it becomes the base.

Problem 18.3

Rewrite using rational exponents.

$$\sqrt[3]{-15ab^2}$$

Solution

Since the index is 3, we know that the denominator will be 3.

$$\sqrt[3]{-15ab^2} = (-15ab^2)^{\frac{1}{3}}$$

Problem 18.4

Rewrite using radical notation. Simplify, if possible.

$$(64)^{\frac{1}{2}}$$

Problem 18.5

Rewrite using radical notation. Simplify, if possible.

$$(-64)^{\frac{1}{3}}$$

Problem 18.6

Rewrite using radical notation. Simplify, if possible.

$$(32)^{\frac{1}{5}}$$

Problem 18.7

Rewrite using radical notation. Simplify, if possible.

$$(7x^2y)^{\frac{1}{4}}$$

Problem 18.8

Rewrite using radical notation. Simplify, if possible.

$$(-13x^4y^2z)^{\frac{1}{5}}$$

Problem 18.9

Rewrite using rational exponents.

$$\sqrt[6]{7x^4y^3z^5}$$

Problem 18.10

Rewrite using rational exponents.

$$\sqrt[7]{-12abc^2}$$

Problem 18.11

Rewrite using rational exponents.

$$\sqrt[3]{\frac{x^2}{y}}$$

Rational exponents can have numerators other than 1. The denominator of the rational exponent becomes the index of the radical expression and the numerator becomes the power.

$$x^{\frac{3}{2}} = (x^{\frac{1}{2}})^3 \text{ or, we could say } x^{\frac{3}{2}} = (x^3)^{\frac{1}{2}}$$

\uparrow
 This becomes $(\sqrt{x})^3$

\uparrow
 This becomes $\sqrt{x^3}$

These two expressions are equivalent

Problem 18.12

Rewrite using radical notation. Simplify, if possible.

$$8^{\frac{2}{3}}$$

Solution

$$8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = (2)^2 = 4$$

Negative Rational Exponent

Just like an integral exponent, a rational exponent can also be negative. Recall that a negative exponent is made positive by moving the base from numerator to denominator (or from denominator to numerator).

$$16^{-\frac{1}{2}} = \frac{1}{16^{\frac{1}{2}}} = \frac{1}{\sqrt{16}} = \frac{1}{4}$$

Problem 18.13

Rewrite using radical notation. Simplify, if possible.

$$32^{\frac{3}{5}}$$

Problem 18.14

Rewrite using radical notation. Simplify, if possible.

$$49^{\frac{5}{2}}$$

Problem 18.15

Rewrite using radical notation. Simplify, if possible.

$$-64^{\frac{2}{3}}$$

Problem 18.16

Rewriting with a positive exponent. Simplify, if possible.

$$32^{-\frac{1}{5}}$$

Problem 18.17

Rewriting with a positive exponent. Simplify, if possible.

$$81^{-\frac{3}{4}}$$

Problem 18.18

Rewriting with a positive exponent. Simplify, if possible.

$$125^{-\frac{2}{3}}$$

Simplifying Expressions with Rational Exponents

We can use the properties of exponents (Section 8) to simplify expressions with rational exponents. For example, when multiplying two terms with the same base, we add the exponents.

$$8^{\frac{1}{9}} \cdot 8^{\frac{4}{9}} = 8^{\frac{1}{9} + \frac{4}{9}} = 8^{\frac{5}{9}}$$

When dividing two terms with the same base, we can subtract the exponents.

$$\frac{24x^{\frac{7}{3}}}{12x^{\frac{2}{3}}} = 2x^{\frac{7}{3} - \frac{2}{3}} = 2x^{\frac{5}{3}}$$

When raising a power to a power, we can multiply the exponents.

$$\left(y^{\frac{1}{3}}\right)^{\frac{2}{5}} = y^{\frac{1}{3} \cdot \frac{2}{5}} = y^{\frac{2}{15}}$$

Problem 18.19

Simplify.

$$7^{\frac{2}{3}} \cdot 7^{\frac{1}{3}}$$

Problem 18.20

Simplify.

$$\frac{28a^{\frac{7}{8}}}{-7a^{\frac{1}{8}}}$$

Problem 18.21

Simplify.

$$\left(x^{\frac{2}{3}}\right)^{\frac{3}{4}}$$

Often, we can use rational exponents to attempt a problem that would be difficult to do in radical form.

College Algebra through Problem Solving by Cifone, Puri, Maslanko, & Dabkowska.



Problem 18.22

Rewrite with rational exponents and simplify. Assume that variables represent positive numbers.

$$\sqrt{x} \cdot \sqrt[3]{x}$$

Solution

Since the index of each expression is not the same, the expressions cannot be multiplied. We will rewrite each expressions using rational exponents then apply the product rule for exponents.

$\sqrt{x} \cdot \sqrt[3]{x}$	Original expression
$= x^{\frac{1}{2}} \cdot x^{\frac{1}{3}}$	Rewrite using rational exponents
$= x^{\frac{1}{2} + \frac{1}{3}}$	By using the product rule, add exponents
$= x^{\frac{3}{6} + \frac{2}{6}}$	Use the LDC to obtain equivalent fractions
$= x^{\frac{5}{6}}$	Simplify

Problem 18.23

Rewrite with rational exponents and simplify. Assume that variables represent positive numbers.

$$\sqrt[5]{32x^{10}}$$

Solution

Once the radical expression is converted to an expression with a rational exponent, the exponent rules can be applied.

$\sqrt[5]{32x^{10}}$	Original expression
$= (32x^{10})^{\frac{1}{5}}$	Rewrite using rational exponents
$= 32^{\frac{1}{5}} x^{10 \cdot \frac{1}{5}}$	By using the product raised to a power rule, distribute the exponent to each factor
$= 2x^{\frac{10}{5}}$	Simplify and apply the power rule for exponents
$= 2x^2$	Simplify further

Problem 18.24

Rewrite with rational exponents and simplify.
Assume that variables represent positive numbers.

$$\sqrt[5]{x} \cdot \sqrt[3]{x}$$

Problem 18.25

Rewrite with rational exponents and simplify.
Assume that variables represent positive numbers.

$$\frac{\sqrt{x}}{\sqrt[3]{x}}$$

Problem 18.26

Rewrite with rational exponents and simplify.
Assume that variables represent positive numbers.

$$\sqrt[3]{64x^9}$$

Problem 18.27

Rewrite with rational exponents and simplify.
Assume that variables represent positive numbers.

$$\sqrt[8]{x^4y^2}$$

Section 19: Simplifying Radical Expressions

A radical expression, whose **index** (or power) is n , is simplified when the radicand has no factors that are greater than perfect n th powers. We can simplify radical expressions by factoring the radicand and using the product rule for radicals.

The Product Rule for Radicals

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}, \text{ where } a, b \geq 0 \text{ and the index } n \text{ remains the same}$$

Problem 19.1

Simplify.

$$3\sqrt{72}$$

Solution

Since there is no index written, the index is understood to be 2. We will start by writing 72 as the product of two factors, one being a perfect *square* (because the index is 2).

$3\sqrt{72}$	Original expression
$3\sqrt{36 \cdot 2}$	36 is the GREATEST perfect square factor of 72
$3\sqrt{36} \cdot \sqrt{2}$	Use the product rule to separate into two radicals
$\underbrace{3 \cdot 6}_{18} \sqrt{2}$	Take the square root of 36
$18\sqrt{2}$	Multiply original and new coefficients

NOTE: You may have found that 72 has other perfect square factors. Choosing the GREATEST perfect square will minimize your steps. Let's look at two other simplifications that cause us to simplify the radicand twice.

$3\sqrt{72}$ $3\sqrt{9} \cdot \sqrt{8}$ Now simplify $\sqrt{8}$ $3 \cdot 3 \sqrt{4} \cdot \sqrt{2}$ $3 \cdot 3 \cdot 2 \sqrt{2}$ $18\sqrt{2}$	$3\sqrt{72}$ $3\sqrt{4} \cdot \sqrt{18}$ Now simplify $\sqrt{18}$ $3 \cdot 2 \sqrt{9} \cdot \sqrt{2}$ $3 \cdot 2 \cdot 3 \sqrt{2}$ $18\sqrt{2}$
---	--

Problem 19.2

Simplify.

$$\sqrt[3]{-32}$$

Solution

In this case, the index is 3 so we will look for a *perfect cube*. -8 is the greatest perfect cube that is a factor of -32 .

$$\begin{aligned}\sqrt[3]{-32} &= \sqrt[3]{-8 \cdot 4} \\ &= \sqrt[3]{-8} \cdot \sqrt[3]{4} \\ &= -2 \sqrt[3]{4}\end{aligned}$$

If there are variables in the radicand, we must factor them as well. When deciding on how to factor the variable, we will **look for the exponent to be a multiple of the index**.

Problem 19.3

Simplify.

$$\sqrt{18x^2y^5}$$

Solution

To simplify, we look for a factor of $18x^2y^5$ that is a *perfect square* (exponents that are a multiple of 2).

$$\begin{aligned}&\sqrt{9x^2y^4 \cdot 2y} \\ &\sqrt{9x^2y^4} \cdot \sqrt{2y} \\ &(9x^2y^4)^{\frac{1}{2}} \cdot \sqrt{2y} \\ &3xy^2\sqrt{2y}\end{aligned}$$

$9, x^2$, and y^4 are all perfect squares; 2 and y are not

Separate into 2 separate radicals

Take the square root of the first radical containing all perfect squares. We can do this by applying rational exponents.

Problem 19.4

Simplify.

$$\sqrt[3]{a^4b^7}$$

Solution

In this case, we look for a factor that is a *perfect cube* (exponents that are multiples of 3)

$$\begin{aligned}\sqrt[3]{a^4b^7} &= \sqrt[3]{a^3b^6} \cdot \sqrt[3]{ab} \\ &= ab^2 \sqrt[3]{ab}\end{aligned}$$

Problem 19.5

Simplify.

$\sqrt{75}$

Problem 19.6

Simplify.

$-5\sqrt{27}$

Problem 19.7

Simplify.

$-2\sqrt[3]{54}$

Problem 19.8

Simplify.

$\sqrt[4]{48a^4}$

Problem 19.9

Simplify.

$\sqrt{50x^8y^9}$

Problem 19.10

Simplify.

$\sqrt[3]{-16m^5n^3}$

Multiplying Radical Expressions

When multiplying radical expressions with the same index, we apply the product rule in the other direction.

The Product Rule for Radicals

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}, \text{ where } a, b \geq 0 \text{ and the index } n \text{ is the same.}$$

Problem 19.11

Multiply and simplify.

$\sqrt{15x} \cdot \sqrt{3x}$

Solution

Since both radical expressions have an index of 2, we can multiply the radicands.

$\sqrt{15x \cdot 3x}$

Use product rule to express under one square root

$\sqrt{45x^2}$

Multiply

$\sqrt{9x^2} \sqrt{5}$

Factor and group perfect square factors

$3x \sqrt{5}$

Simplify

Problem 19.12

Multiply and simplify.

$$\sqrt{10ab} \cdot \sqrt{5a^3}$$

Problem 19.13

Multiply and simplify.

$$(-2\sqrt{6})(3\sqrt{14x})$$

Problem 19.14

Multiply and simplify.

$$\sqrt[3]{-3x^2y} \cdot \sqrt[3]{6xy^2}$$

Problem 19.15

Multiply and simplify.

$$\sqrt[4]{8m^2} \cdot \sqrt[4]{2m^3}$$

Adding and Subtracting Radical Expressions

Two (or more) radical expressions that have **the same index AND the same radicand** can be added or subtracted. These expressions are referred to as *like radicals* and can be combined in the same way we combine like terms involving variables.

Problem 19.16

Simplify by combining like radicals.

$$4\sqrt{3} + 7\sqrt{3}$$

Solution

Since both terms have an index of 2 and a radicand of 3, we can add the expressions. To do so, add the coefficients and keep the radicand the same.

$$4\sqrt{3} + 7\sqrt{3} = (4 + 7)\sqrt{3} = 11\sqrt{3}$$

Very often, we are presented with an expression that does not seem to contain like radicals. In these cases, we must first simplify the radicands then combine any like radicals.

Problem 19.17

Simplify by combining like radicals.

$$-9\sqrt{12x} + 7\sqrt{27x}$$

Solution

The original expression does not contain like radicals so we must simplify.

$-9\sqrt{12x} + 7\sqrt{27x}$	Original expression
$-9\sqrt{4} \cdot \sqrt{3x} + 7\sqrt{9} \cdot \sqrt{3x}$	Factor the radicand using a perfect square factor
$-9 \cdot 2\sqrt{3x} + 7 \cdot 3\sqrt{3x}$	Take the square root of the perfect squares
$-18\sqrt{3x} + 21\sqrt{3x}$	Multiply the coefficients
$3\sqrt{3x}$	Combine like radicals

Problem 19.18

Simplify by combining like radicals.

$$-3\sqrt{5x} + 10\sqrt{5x}$$

Problem 19.19

Simplify by combining like radicals.

$$5\sqrt[3]{7} + 4\sqrt[3]{2} - \sqrt[3]{7}$$

Problem 19.20

Simplify by combining like radicals.

$$2\sqrt{18} + 3\sqrt{32} - 16\sqrt{2}$$

Problem 19.21

Simplify by combining like radicals.

$$5\sqrt[3]{16x^2} - 6\sqrt[3]{54x^2}$$

Multiplying Multiple Radical Expressions

Just as with adding and subtracting, multiplying radical expression with more than one term follows similar concepts as multiplying polynomial expression. We can distribute and monomial radical expression and even “double distribute” a binomial times a binomial.

Problem 19.22

Multiply and simplify.

$$\sqrt{3}(\sqrt{6} + 5)$$

Solution

To multiply, we will distribute $\sqrt{3}$ to the two terms in the parentheses.

$\sqrt{3}(\sqrt{6} + 5)$	Original expression
$\sqrt{3}(\sqrt{6}) + \sqrt{3}(5)$	Use the product rule to multiply the two radicals in the first term.
$\sqrt{18} + 5\sqrt{3}$	Simplify. (The whole number 5 becomes the coefficient of the second term)
$\sqrt{9}\sqrt{2} + 5\sqrt{3}$	Use the product rule to simplify $\sqrt{18}$
$3\sqrt{2} + 5\sqrt{3}$	Simplify

Problem 19.23

Multiply and simplify.

$$(4\sqrt{2} + \sqrt{5})(\sqrt{2} - 3\sqrt{5})$$

Solution

To multiply a binomial by a binomial, we must multiple each term in the first binomial by each term in the second binomial. You may have heard this method referred to as the FOIL method.

$(4\sqrt{2} + \sqrt{5})(\sqrt{2} - 3\sqrt{5})$	Distribute each of the terms
$4\sqrt{2}(\sqrt{2}) - 4\sqrt{2}(3\sqrt{5}) + \sqrt{5}(\sqrt{2}) - \sqrt{5}(3\sqrt{5})$	Multiply coefficients and radicands in each term
$4\sqrt{4} - 12\sqrt{10} + \sqrt{10} - 3\sqrt{25}$	Simplify perfect square radicands and combine like radicals
$4 \cdot 2 - 11\sqrt{10} - 3 \cdot 5$	
$8 - 11\sqrt{10} - 15$	Multiply integers
$-7 - 11\sqrt{10}$	Simplify (combine integers)

Problem 19.24

Multiply and simplify.

$$\sqrt{7}(x - \sqrt{6})$$

Problem 19.25

Multiply and simplify.

$$\sqrt[3]{2}(\sqrt[3]{5} + 3\sqrt[3]{4})$$

Problem 19.26

Multiply and simplify.

$$(7 + \sqrt{6})(5 - \sqrt{6})$$

Problem 19.27

Multiply and simplify.

$$(\sqrt{3} - \sqrt{10})^2$$

Problem 19.28

Multiply and simplify.

$$(2\sqrt{2} - \sqrt{5})(6\sqrt{2} - 7\sqrt{5})$$

Problem 19.29

Multiply and simplify.

$$(\sqrt{5} - \sqrt{3})(\sqrt{5} + \sqrt{3})$$

Looking closely and **Problem 19.29**, we see that something special happened after simplifying. When multiplying the sum and difference of a binomial containing radical expressions, not only to the middle terms cancel, but the result is an integer. These binomials are called **conjugate pairs**.

Multiplying Radical Conjugates

$$(\sqrt{A} + \sqrt{B})(\sqrt{A} - \sqrt{B}) = A - B$$

When simplified, the product will not contain radicals.

Problem 19.30

Determine the conjugate pair and multiply.

$$(\sqrt{2} - \sqrt{7})$$

Solution

Since this binomial contains subtraction, the conjugate will contain addition and is $(\sqrt{2} + \sqrt{7})$. It is only necessary to multiply the first terms in each binomial and the last terms.

$$\begin{aligned} & (\sqrt{2} - \sqrt{7})(\sqrt{2} + \sqrt{7}) \\ & \quad \swarrow \quad \searrow \\ & \sqrt{2} \cdot \sqrt{2} - \sqrt{7} \cdot \sqrt{7} \\ & \quad \sqrt{4} - \sqrt{49} \\ & \quad 2 - 7 \\ & \quad -5 \end{aligned}$$

Multiplying $\sqrt{2}$ by itself will give $(\sqrt{2})^2 = \sqrt{2} \cdot \sqrt{2}$

Multiplying $\sqrt{7}$ by itself will give $(\sqrt{7})^2 = \sqrt{7} \cdot \sqrt{7}$

Use product rule to multiply radicals

Simplify the perfect square radicals

Simplify

Problem 19.31

Determine the conjugate pair and multiply.

$$(\sqrt{6} + 5\sqrt{2})$$

Problem 19.32

Determine the conjugate pair and multiply.

$$(7 - \sqrt{5})$$

We will see a rather important use for conjugate pairs in following sections.

Section 20: Rationalizing Denominators

A radical expression is not considered completely simplified if there is a radical in the denominator. This process of changing the denominator so that it does not contain any radicals is called **rationalizing the denominator**.

We know that raising a term to a power and taking the root of that power are reverse operations. We have seen examples such as:

$$\begin{aligned}\sqrt{9} &= \sqrt{3^2} = 3 \\ \sqrt[3]{125} &= \sqrt[3]{5^3} = 5 \\ \sqrt[4]{16} &= \sqrt[4]{2^4} = 2\end{aligned}$$

Notice that after we perform the reverse operation, there is no radical sign left, just a rational number. We can use this idea when rationalizing denominators.

Rationalizing Denominators Containing One Term

Problem 20.1

Rationalize the denominator

$$\frac{\sqrt{3}}{\sqrt{13}}$$

Solution

If we multiply the denominator, $\sqrt{13}$ by $\sqrt{13}$, then the denominator will become a rational number.

$$\sqrt{13} \cdot \sqrt{13} = \sqrt{13^2} = 13.$$

We can successfully eliminate the radical in the denominator using this method. However, whatever we do to the denominator we must do to the numerator.

$$\frac{\sqrt{3}}{\sqrt{13}} \cdot \frac{\sqrt{13}}{\sqrt{13}} = \frac{\sqrt{3 \cdot 13}}{\sqrt{13^2}} = \frac{\sqrt{39}}{13}$$

Notice that we are
multiplying by 1 here.

Problem 20.2

Rationalize the denominator

$$\frac{7}{\sqrt[3]{9}}$$

Solution

In Problem 20.1 we needed to force the radicand of the denominator to be a *perfect square*. For this problem we need force the radicand of the denominator to be a *perfect cube*.

Examining the denominator, we see that $\sqrt[3]{9} = \sqrt[3]{3^2}$. If we multiply $\sqrt[3]{3^2}$ by another factor of $\sqrt[3]{3}$ then we will have a radicand that is a perfect cube: $\sqrt[3]{3^2} \cdot \sqrt[3]{3} = \sqrt[3]{3^3} = 3$. Again, we must treat the numerator the same as the denominator:

$$\frac{7}{\sqrt[3]{9}} = \frac{7}{\sqrt[3]{3^2}} \cdot \frac{\sqrt[3]{3}}{\sqrt[3]{3}} = \frac{7\sqrt[3]{3}}{\sqrt[3]{3^3}} = \frac{7\sqrt[3]{3}}{3}$$

Problem 20.3

Rationalize the denominator

$$\frac{\sqrt{2}}{\sqrt{7}}$$

Problem 20.4

Rationalize the denominator

$$\sqrt{\frac{4}{5}}$$

Problem 20.5

Rationalize the denominator

$$\frac{3}{\sqrt{x}}$$

Problem 20.6

Rationalize the denominator

$$\frac{\sqrt[3]{7}}{\sqrt[3]{16}}$$

Rationalizing Denominators that Contain Two Terms

In Section 19, we saw that when we multiply the sum and difference of two terms (or conjugates) that contain square roots, the radicals will be eliminated. This is the methodology that we will use for rationalizing denominators with two terms. We multiply the denominator by its conjugate using:

$$(A + B)(A - B) = A^2 - B^2$$

The resulting product will not contain a radical and the denominator would be rationalized.

Problem 20.7

Rationalize the denominator

$$\frac{6}{3 - \sqrt{2}}$$

Solution

The conjugate of the denominator is $3 + \sqrt{2}$. We must multiply the numerator and the denominator by the conjugate to rationalize the denominator.

$$\begin{aligned} \frac{6}{3 - \sqrt{2}} \cdot \frac{3 + \sqrt{2}}{3 + \sqrt{2}} \\ &= \frac{6(3 + \sqrt{2})}{(3)^2 - (\sqrt{2})^2} \\ &= \frac{6(3 + \sqrt{2})}{9 - 2} \\ &= \frac{6(3 + \sqrt{2})}{7} \end{aligned}$$

Problem 20.8

Rationalize the denominator

$$\frac{5 - \sqrt{2}}{7 + \sqrt{3}}$$

Solution

The conjugate of the denominator is $7 - \sqrt{3}$. We need to multiply the numerator and the denominator by the conjugate to rationalize the denominator.

$$\begin{aligned} \frac{5 - \sqrt{2}}{7 + \sqrt{3}} \cdot \frac{7 - \sqrt{3}}{7 - \sqrt{3}} \\ &= \frac{(5 - \sqrt{2})(7 - \sqrt{3})}{(7)^2 - (\sqrt{3})^2} \end{aligned}$$

$$= \frac{(5)(7) - 5\sqrt{3} - 7\sqrt{2} + (\sqrt{2})(\sqrt{3})}{49 - 3}$$

$$= \frac{35 - 5\sqrt{3} - 7\sqrt{2} + \sqrt{6}}{46}$$

Problem 20.9

Rationalize the denominator: $\frac{7}{\sqrt{5}-2}$

Problem 20.10

Rationalize the denominator: $\frac{3}{11+\sqrt{7}}$

Problem 20.11

Rationalize the denominator: $\frac{8+\sqrt{3}}{6-\sqrt{2}}$

Problem 20.12

Rationalize the denominator: $\frac{3-\sqrt{6}}{\sqrt{7}+5}$

Problem 20.13

Rationalize the denominator: $\frac{11}{\sqrt{6}-\sqrt{2}}$

Problem 20.14

Rationalize the denominator: $\frac{3}{\sqrt{7}+\sqrt{5}}$

Problem 20.15

Rationalize the denominator: $\frac{\sqrt{7}-\sqrt{3}}{\sqrt{7}+\sqrt{3}}$

Problem 20.16

Rationalize the denominator: $\frac{\sqrt{11}+\sqrt{3}}{\sqrt{11}-\sqrt{3}}$

Section 21: Radical Equations

We are ready to solve equations that involve radical expressions. We recall the following fact:

$$(\sqrt[n]{x})^n = x$$

So, if we wish to eliminate a radical, we must raise it to the power of the index. For example,

$$(\sqrt{5})^2 = 5$$

$$(\sqrt[3]{7})^3 = 7$$

$$(\sqrt[7]{15})^7 = 15 \text{ etc.}$$

We will use this fact to help us in solving the following equations with radical expressions.

Problem 21.1

Solve for x and check.

$$\sqrt{2x+1} = 3$$

Solution

Since we have a square root here, we must remove it by squaring. However, we must remember to perform the same operation on both sides of the equation. Therefore, we square both sides!

$$(\sqrt{2x+1})^2 = (3)^2$$

$$2x + 1 = 9$$

$$2x = 8$$

$$x = 4$$

Square both sides.

Eliminate the radical and simplify.

Subtract 1 from each side.

Divide by 2 and simplify.

Now, we must check our answer by substituting $x = 4$ into the original equation.

$$\sqrt{2x+1} = 3$$

$$\sqrt{2(4)+1} \stackrel{?}{=} 3$$

$$\sqrt{8+1} \stackrel{?}{=} 3$$

$$\sqrt{9} \stackrel{?}{=} 3$$

$$3 = 3 \quad \checkmark$$

Problem 21.2Solve for x and check.

$$\sqrt{2x+3} = x$$

Solution

We proceed as before.

$$(\sqrt{2x+3})^2 = (x)^2$$

$$2x + 3 = x^2$$

$$0 = x^2 - 2x - 3$$

$$0 = x^2 - 2x - 3$$

$$0 = (x-3)(x+1)$$

$$x = 3, \quad x = -1$$

Square both sides.

Eliminate the radical.

Make the left hand side zero.

Factor the quadratic expression.

Solve by setting each factor equal to zero.

Now, we must check our answer by substituting $x = 3$ and $x = -1$ into the original equation.

$$\sqrt{2x+3} = x$$

$$\sqrt{2(3)+3} \neq 3$$

$$\sqrt{6+3} \neq 3$$

$$\sqrt{9} \neq 3$$

$$3 = 3 \quad \checkmark$$

$$\sqrt{2x+3} = x$$

$$\sqrt{2(-1)+3} \neq -1$$

$$\sqrt{-2+3} \neq -1$$

$$\sqrt{1} \neq -1$$

$$1 \neq -1$$

Therefore, we see that only $x = 3$ is a solution.**Problem 21.3**Solve for x and check.

$$\sqrt{4x-8} + 2 = x$$

SolutionHere, we must make a small adjustment before squaring. We must **isolate the radical**. In order to do this, we must subtract 2 from each side. Therefore, we have the following.

$$\sqrt{4x-8} = x-2$$

$$(\sqrt{4x-8})^2 = (x-2)^2$$

$$4x - 8 = x^2 - 4x + 4$$

$$0 = x^2 - 8x + 12$$

$$0 = (x-6)(x-2)$$

$$x = 6, \quad x = 2$$

Square both sides.

Eliminate the radical and expand the right side.

Make the left hand side zero.

Factor the quadratic expression.

Solve by setting each factor equal to zero.

Now, we must check our answer by substituting $x = 6$ and $x = 2$ into the original equation.

$$\begin{aligned}\sqrt{4x - 8} + 2 &= x \\ \sqrt{4(6) - 8} + 2 &\neq 6 \\ \sqrt{24 - 8} + 2 &\neq 6 \\ \sqrt{16} + 2 &\neq 6 \\ 4 + 2 &\neq 6 \\ 6 &= 6 \quad \checkmark\end{aligned}$$

$$\begin{aligned}\sqrt{4x - 8} + 2 &= x \\ \sqrt{4(2) - 8} + 2 &\neq 2 \\ \sqrt{8 - 8} + 2 &\neq 2 \\ \sqrt{0} + 2 &\neq 2 \\ 0 + 2 &\neq 2 \\ 2 &= 2 \quad \checkmark\end{aligned}$$

Therefore, we have two solutions, namely $x = 6$, and $x = 2$.

Problem 21.4

Solve for x and check.

$$\sqrt{9x - 8} = 8$$

Problem 21.5

Solve for x and check.

$$\sqrt{3x - 6} + 2 = 5$$

Problem 21.6Solve for x and check.

$$\sqrt{6x + 16} = x$$

Problem 21.7Solve for x and check.

$$\sqrt{5x + 19} + 1 = x$$

Cube roots and equations with two radical expressions

Let us explore a situation where we encounter the cube root.

Problem 21.8Solve for x and check.

$$\sqrt[3]{7x - 1} = 3$$

Solution

Now, since we must remove the cube root, we proceed by raising both sides to the third power.

$$(\sqrt[3]{7x - 1})^3 = (3)^3$$

$$7x - 1 = 27$$

$$7x = 28$$

$$x = 4$$

Cube both sides.

Eliminate the radical and simplify.

Add 1 to each side.

Divide by 7 and simplify.

Now, we must check our answer by substituting $x = 4$ into the original equation.

$$\sqrt[3]{7x - 1} = 3$$

$$\sqrt[3]{7(4) - 1} \neq 3$$

$$\sqrt[3]{28 - 1} \neq 3$$

$$\sqrt[3]{27} \neq 3$$

$$3 = 3 \quad \checkmark$$

Therefore, our solution is $x = 4$.

What about a situation where we encounter two radical expressions? For example,

Problem 21.9

Solve for x and check.

$$\sqrt{5x + 4} = \sqrt{7x - 20}$$

Solution

Here, we simply square both sides to remove both radicals simultaneously!

$$(\sqrt{5x + 4})^2 = (\sqrt{7x - 20})^2$$

$$5x + 4 = 7x - 20$$

Now, solve this equation and check your answer.

Problem 21.10Solve for x and check.

$$\sqrt[3]{5x + 14} = 4$$

Problem 21.11Solve for x and check.

$$\sqrt{3x + 1} = \sqrt{5x - 15}$$

Problem 21.12Solve for x and check.

$$\sqrt[3]{3x - 4} = 2$$

Problem 21.13Solve for x and check.

$$\sqrt{2x + 16} + 4 = x$$

Section 22: Complex Numbers

Now that we are familiar with roots and radicals, we are ready to explore the world of complex numbers. Let us compare the following two examples.

Problem 22.1

Solve for x .

$$x^2 - 1 = 0$$

Solution

$$\begin{aligned}(x - 1)(x + 1) &= 0 \\ x - 1 = 0 \text{ or } x + 1 &= 0 \\ x = 1 \text{ or } x &= -1\end{aligned}$$

Factor the difference of two squares.
Set each factor equal to zero.
Solve.

The above problem was relatively easy to solve. We make a small adjustment and pose the next problem.

Problem 22.2

Solve for x .

$$x^2 + 1 = 0$$

Solution

Notice that here the addition makes it impossible to factor. Instead, we use the only other logical technique available, and isolate the x -term.

$$\begin{aligned}x^2 + 1 - 1 &= 0 - 1 \\ x^2 &= -1 \\ \sqrt{x^2} &= \pm\sqrt{-1} \\ x &= \pm\sqrt{-1}\end{aligned}$$

Subtract 1 from each side.
Simplify.
Take the square root of x^2 to get x .
Simplify.

Now, we are left with an unfamiliar situation. We know that we cannot take the square root of a negative number. To get around this problem, we will define a new number as follows.

Definition

We define a new, non-real number called i such that

$$i = \sqrt{-1}$$

It follows that $i^2 = -1$.

Therefore, to solve our equation in Problem 22.2, we finally replace $\sqrt{-1}$ with i , to get $x = \pm i$.

■

Problem 22.3

Evaluate the following.

- a. $\sqrt{-9}$
- b. $\sqrt{-25}$
- c. $\sqrt{-7}$
- d. $\sqrt{-8}$

Solution

For all these questions, we proceed as we did in Section 19 with the product rule for radicals

- a. $\sqrt{-9} = \sqrt{9 \cdot -1} = \sqrt{9}\sqrt{-1} = 3i.$
- b. $\sqrt{-25} = \sqrt{25 \cdot -1} = \sqrt{25}\sqrt{-1} = 5i.$
- c. $\sqrt{-7} = \sqrt{7 \cdot -1} = \sqrt{7}\sqrt{-1} = \sqrt{7}i.$
- d. $\sqrt{-8} = \sqrt{8 \cdot -1} = \sqrt{4 \cdot 2 \cdot -1} = \sqrt{4}\sqrt{2}\sqrt{-1} = 2\sqrt{2}i.$



Now that we have defined i , we can treat it just like any number or variable, thus allowing us to perform all the operations of addition, subtraction, multiplication and division with and by i .

In fact, we have now expanded our set of numbers to include numbers like $2i$, $-7i$, $1 + i$, $3 - 4i$ etc. It is time for us to generalize this concept.

Complex Numbers

A complex number is a number of the form

$$a + bi$$

Here, a and b are ordinary real numbers and $i = \sqrt{-1}$.

We refer to a as the **real part**, and to b as the **imaginary part** of the complex number.

Problem 22.4

Add and express the answer in $a + bi$ form.

$$(2 - 3i) + (-7 + 4i)$$

Solution:

We treat this just as any addition with variable terms. We add like terms.

$$\begin{aligned} & (2 - 3i) + (-7 + 4i) \\ &= (2 + -7) + (-3i + 4i) \\ &= -5 + i \end{aligned}$$

Problem 22.5

Subtract and express the answer in $a + bi$ form.

$$(7 + 9i) - (3 + 16i)$$

Solution

We treat this just as any subtraction with variable terms. We change it to an addition first, changing all the signs in the second expression, and then we add like terms.

$$\begin{aligned}(7 + 9i) - (3 + 6i) &= (7 + 9i) + (-3 - 6i) \\ &= (7 + -3) + (9i + -16i) \\ &= 4 - 7i\end{aligned}$$

Problem 22.6

Evaluate.

$$\sqrt{-49}$$

Problem 22.7

Evaluate and simplify.

$$\sqrt{-90}$$

Problem 22.8

Add and express the answer in $a + bi$ form.

$$(4 - i) + (-12 + 14i)$$

Problem 22.9

Subtract and express the answer in $a + bi$ form.

$$(-22 - 13i) - (-17 + 8i)$$

Problem 22.10

Multiply and express the answer in $a + bi$ form.

$$(2 - 6i)(7 + 3i)$$

Solution

$$\begin{aligned}
 & (2 - 6i)(7 + 3i) \\
 &= 2(7 + 3i) - 6i(7 + 3i) && \text{Distribute.} \\
 &= 14 + 6i - 42i - 18i^2 && \text{Simplify.} \\
 &= 14 - 36i - 18 \cdot (-1) && \text{Combine like-terms and replace } i^2 \text{ by } -1. \\
 &= 14 - 36i + 18 && \text{Simplify.} \\
 &= 32 - 36i && \text{Write the final answer in } a + bi \text{ form.}
 \end{aligned}$$

Problem 22.11

Divide and express the answer in $a + bi$ form.

$$\frac{4 - 3i}{5i}$$

Solution

Here, we use the fact that $i^2 = -1$ to our advantage. Notice the similarity with **rationalizing the denominators in Section 20**.

$$\begin{aligned}
 & \frac{4 - 3i}{5i} \\
 &= \frac{(4 - 3i)}{5i} \cdot \frac{i}{i} && \text{Multiply numerator and denominator by } i. \\
 &= \frac{4i - 3i^2}{5i^2} && \text{Distribute.} \\
 &= \frac{4i - 3 \cdot (-1)}{5 \cdot (-1)} && \text{Replace } i^2 \text{ by } -1. \\
 &= \frac{4i + 3}{-5} && \text{Simplify.} \\
 &= \frac{4i}{-5} + \frac{3}{-5} && \text{Split into two separate fractions.} \\
 &= -\frac{3}{5} - \frac{4}{5}i && \text{Rearrange to write in } a + bi \text{ form.}
 \end{aligned}$$

Problem 22.12

Divide and express the answer in $a + bi$ form.

$$\frac{3 - 5i}{7 + 6i}$$

Solution

Since we have two terms in the denominator, we now multiply the numerator and denominator by the conjugate of the denominator, just as in section 20.

$$\begin{aligned} & \frac{3 - 5i}{7 + 6i} \\ &= \frac{(3 - 5i) \cdot (7 - 6i)}{(7 + 6i) \cdot (7 - 6i)} \\ &= \frac{3(7 - 6i) - 5i(7 - 6i)}{(7)^2 - (6i)^2} \\ &= \frac{21 - 18i - 35i + 30i^2}{49 - 36i^2} \\ &= \frac{21 - 53i + 30 \cdot (-1)}{49 - 36 \cdot (-1)} \\ &= \frac{21 - 53i - 30}{49 + 36} \\ &= \frac{-9 - 53i}{85} \\ &= -\frac{9}{85} - \frac{53}{85}i \end{aligned}$$

Multiply numerator and denominator by the conjugate of the denominator.

Distribute and notice that the denominator is just the difference of two squares.

Expand.

Combine like-terms and replace i^2 by -1 .

Simplify further.

Combine like-terms.

Split into two fractions and write in $a + bi$ form.

Problem 22.13

Multiply and express the answer in $a + bi$ form.

$$(3 - 4i)(2 + 3i)$$

Problem 22.14

Multiply and express the answer in $a + bi$ form.

$$i(5 - 6i)$$

Problem 22.15Expand and express the answer in $a + bi$ form.

$$(5 + 2i)^2$$

Problem 22.16Multiply and express the answer in $a + bi$ form.

$$(3 - 2i)(3 + 2i)$$

Problem 22.17Divide and express the answer in $a + bi$ form.

$$\frac{7 + 8i}{3i}$$

Problem 22.18Divide and express the answer in $a + bi$ form.

$$\frac{4i}{2 + 5i}$$

Problem 22.19Divide and express the answer in $a + bi$ form.

$$\frac{3 + 4i}{3 - 4i}$$

Problem 22.20Divide and express the answer in $a + bi$ form.

$$\frac{8 + 3i}{-2 - 5i}$$

Section 23: Completing the Square

We will be revisiting the world of quadratic equations. Our task will be to solve any quadratic equation, whether it is factorable or not. For equations that are not factorable, we use a technique called completing the square. Before we explore this, let us discuss an important property.

The Square Root Property

If $x^2 = b$, then we may take the square root on both sides to get $x = \pm\sqrt{b}$

As a result, if $(x + a)^2 = b$, then $x + a = \pm\sqrt{b}$

Problem 23.1

Solve the following equation by using the square root property.

$$(x + 2)^2 = 7$$

Solution

Since the variable is part of a binomial that is squared, we will start by applying the square root property to remove the exponent. Remember to account for the positive and negative square root.

$(x + 2)^2 = 7$	Original equation
$\sqrt{(x + 2)^2} = \pm\sqrt{7}$	Take the square root of both sides
$x + 2 = \pm\sqrt{7}$	Simplify
$x = -2 \pm \sqrt{7}$	Solve for x by subtracting 2 from both sides

In the remaining problems, we will see how the technique of creating a **perfect square trinomial** will lead to the square root property. A perfect square trinomial is a trinomial that will factor into a binomial squared. We will first practice creating a perfect square trinomial before solving another equation.

Perfect Square Trinomials

$$a^2 + 2ab + b^2 = (a + b)^2$$

$$\text{For example, } x^2 + 6x + 9 = (x + 3)^2$$

$$x^2 - 2x + 1 = (x - 1)^2$$

Problem 23.2

Complete the square to create a perfect square trinomial. Then factor.

$$x^2 - 8x$$

Solution

To complete the square, we add a constant to a binomial that is in the form $x^2 + bx$. To find that constant, we will half the coefficient of x then square that value $\left(\frac{b}{2}\right)^2$

$$x^2 - 8x$$

Original expression

$$\left(\frac{b}{2}\right)^2 = \left(\frac{-8}{2}\right)^2 = (-4)^2 = 16$$

Determine the constant needed to create a perfect square trinomial.

$$x^2 - 8x + 16$$

Add the constant to the original binomial.

$$(x - 4)(x - 4)$$

Factor the perfect square trinomial. The result will be a binomial squared.

Problem 23.3

Solve the following equation by using the square root property.

$$(x - 3)^2 = 15$$

Problem 23.4

Solve the following equation by using the square root property.

$$(x + 6)^2 = 20$$

Problem 23.5

Complete the square to create a perfect square trinomial. Then factor.

$$x^2 - 6x$$

Problem 23.6

Complete the square to create a perfect square trinomial. Then factor.

$$x^2 + 10x$$

Problem 23.7

Solve the following equation by completing the square.

$$x^2 + 6x + 2 = 0$$

Solution

Notice that the expression above is not factorable. We proceed by completing the square below.

$$x^2 + 6x + 2 = 0$$

The leading coefficient must be 1. If not, then divide each term by the leading coefficient.

$$x^2 + 6x + 2 - 2 = 0 - 2$$

Remove the constant term from the left side to leave a binomial in the form $x^2 + bx$.

$$x^2 + 6x = -2$$

$$x^2 + 6x + 9 = -2 + 9$$

Determine the constant to each side that will complete the square. $\left(\frac{b}{2}\right)^2 = \left(\frac{6}{2}\right)^2 = 9$.

$$(x + 3)^2 = 7$$

Factor the perfect square trinomial.

$$x + 3 = \pm\sqrt{7}$$

Use the square root property.

$$x + 3 - 3 = \pm\sqrt{7} - 3$$

Subtract 3 from each side.

$$x = -3 \pm \sqrt{7}$$

Rearrange to write the radical in the end.

$$x = -3 + \sqrt{7}, \quad x = -3 - \sqrt{7}$$

Split to reveal both solutions.

In some cases, the leading coefficient may be some value other than 1. We must first divide all terms by the leading coefficient before proceeding to complete the square.

Problem 23.8

Solve the following equation by completing the square.

$$2x^2 - 8x + 12 = 0$$

Solution

$$\frac{2x^2}{2} - \frac{8x}{2} + \frac{12}{2} = \frac{0}{2}$$

Make the leading coefficient 1 by dividing each term of the equation by 2.

$$x^2 - 4x + 6 = 0$$

Simplify.

$$x^2 - 4x + 6 - 6 = 0 - 6$$

Remove the constant term from the left side.

$$x^2 - 4x = -6$$

$$x^2 - 4x + 4 = -6 + 4$$

$$(x - 2)^2 = -2$$

$$x - 2 = \pm\sqrt{-2}$$

$$x - 2 = \pm i\sqrt{2}$$

$$x - 2 + 2 = \pm i\sqrt{2} + 2$$

$$x = 2 \pm i\sqrt{2}$$

$$x = 2 + i\sqrt{2}, \quad x = 2 - i\sqrt{2}$$

Problem 23.9

Solve the following equation by completing the square.

$$x^2 + 8x - 10 = 0$$

Determine the constant to each side that will complete the square. $\left(\frac{b}{2}\right)^2 = \left(\frac{-4}{2}\right)^2 = 4$.

Factor the trinomial to complete the square.

Use the square root property.

Use the fact that $\sqrt{-1} = i$.

Subtract 2 from each side.

Rearrange to write the radical in the end.

Split to reveal both solutions.

Problem 23.10

Solve the following equation by completing the square.

$$x^2 + 10x + 5 = 0$$

Problem 23.11

Solve the following equation by using the square root property.

$$(x + 1)^2 = -8$$

Problem 23.12

Solve the following equation by completing the square.

$$x^2 + 2x + 10 = 0$$

Problem 23.13

Solve the following equation by completing the square.

$$2x^2 - 12x + 24 = 0$$

Problem 23.14

Solve the following equation by completing the square.

$$x^2 + 5x + 1 = 0$$

Section 24: The Quadratic Formula

There is another way of solving quadratic equations, whether they are factorable or not. If we take the standard form of a quadratic equation, $ax^2 + bx + c = 0$, and use completing the square to solve for x , we arrive at the **quadratic formula**.

A quadratic equation of the form $ax^2 + bx + c = 0$, with $a \neq 0$, has solutions given by the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The first thing to do when using the quadratic formula is to make sure that the given equation is in standard form. This is very important in order to determine the correct values for a , b and c to substitute into the formula. The \pm sign in front of $\sqrt{b^2 - 4ac}$ lets us know that there are two solutions to the equation, just as with factoring and completing the square. These solutions may not be distinct.

Problem 24.1

Solve using the quadratic formula: $6x^2 + 3x - 1 = 0$

Solution

Since this equation is already in standard form, we can identify the a , b and c .

$$6x^2 + 3x - 1 = 0$$

\uparrow
 $a=6$

\uparrow
 $b=3$

\uparrow
 $c=-1$

Now we can substitute these values into the quadratic formula and solve for x .

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

List the quadratic formula.

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(6)(-1)}}{2(6)}$$

Substitute in the values for a , b and c .
In this case, $a = 6$, $b = 3$, $c = -1$.

$$x = \frac{-3 \pm \sqrt{9 - (-24)}}{12}$$

Simplify as much as possible.

$$x = \frac{-3 \pm \sqrt{9 + 24}}{12}$$

$$x = \frac{-3 \pm \sqrt{33}}{12}$$

Notice that we now have two solutions:

$$x = \frac{-3 + \sqrt{33}}{12} \text{ and } x = \frac{-3 - \sqrt{33}}{12}$$

(These solutions are *irrational conjugates*.)

Notice that in the problem above, you might be tempted to try to simplify further. After all, $\frac{-3}{12}$ could be simplified. However, as a reminder, *you cannot divide just one term in the numerator by the denominator.*

$$x = \frac{-3 \pm \sqrt{33}}{12} \neq \frac{-1 \pm \sqrt{33}}{4}$$

Problem 24.2

Solve using the quadratic formula: $2x^2 = -9x + 5$

Solution

This equation is not in standard form. We need to set the equation equal to zero so that we can identify the values of a , b and c .

$$\begin{aligned} 2x^2 + 9x &= -9x + 9x + 5 \\ 2x^2 + 9x - 5 &= +5 - 5 \\ 2x^2 + 9x - 5 &= 0 \end{aligned}$$

\nwarrow
 $a=2$

\nwarrow
 $b=9$

\nwarrow
 $c=-5$

Now we can substitute these values into the quadratic formula and solve for x .

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

List the quadratic formula.

$$x = \frac{-9 \pm \sqrt{(9)^2 - 4(2)(-5)}}{2(2)}$$

Substitute in the values for a , b and c .

In this case, $a = 2$, $b = 9$, $c = -5$.

$$x = \frac{-9 \pm \sqrt{81 - (-40)}}{4}$$

Simplify as much as possible.

$$x = \frac{-9 \pm \sqrt{81 + 40}}{4}$$

$$x = \frac{-9 \pm \sqrt{121}}{4}$$

$$x = \frac{-9 \pm 11}{4}$$

We now have two solutions:

$$x = \frac{-9 + 11}{4} = \frac{2}{4} = \frac{1}{2}$$

and

$$x = \frac{-9 - 11}{4} = \frac{-20}{4} = -5$$

So the solutions are $x = \frac{1}{2}$ and $x = -5$.

The last two problems had *real solutions*. We learned about complex numbers in Section 22. It is possible to have a complex number as solution. This will happen when the expression $b^2 - 4ac$ (also known as the *discriminant*) yields a negative number.

Problem 24.3

Solve using the quadratic formula: $3x^2 - 2x = -7$

Solution

This equation is not in standard form. We need to set the equation equal to zero so that we can identify the a , b and c .

$$\begin{aligned}
 3x^2 - 2x &= -7 \\
 3x^2 - 2x + 7 &= -7 + 7 \\
 3x^2 - 2x + 7 &= 0
 \end{aligned}$$

\swarrow \swarrow \swarrow
 $a=3$ $b=-2$ $c=7$

Now we can substitute these values into the quadratic formula and solve for x .

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

List the quadratic formula.

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(7)}}{2(3)}$$

Substitute in the values for a , b and c .
In this case, $a = 3$, $b = -2$, $c = 7$.

$$x = \frac{4 \pm \sqrt{4 - 84}}{6}$$

Simplify as much as possible.

$$x = \frac{2 \pm \sqrt{-80}}{6} = \frac{2 \pm \sqrt{16 \cdot 5 \cdot (-1)}}{6}$$

$$x = \frac{2 \pm 4i\sqrt{5}}{6}$$

$$x = \frac{2 \pm 4i\sqrt{5}}{6} \div \frac{2}{2} = \frac{1 \pm 2i\sqrt{5}}{3}$$

We now have two solutions:

$$x = \frac{1+2i\sqrt{5}}{3} \quad \text{and} \quad x = \frac{1-2i\sqrt{5}}{3}$$

Notice that these solutions are *complex conjugates*. This will always be the case when the solutions are complex numbers.

Now let's practice using the quadratic formula with the following problems. ***It is good practice to write the quadratic formula when you start each problem to help you substitute correctly and to help remember it!***

Problem 24.4

Solve using the quadratic formula.

$$6x^2 - 9x - 7 = 0$$

Problem 24.5

Solve using the quadratic formula.

$$x^2 + 7x = -10$$

Problem 24.6

Solve using the quadratic formula.

$$4x^2 = 6x - 3$$

Problem 24.7

Solve using the quadratic formula.

$$3x^2 = -2x + 2$$

In Section 12 we solved word problems that involved quadratic equations using the formula for the area of a rectangle. Those equations were factorable. The problems below are not factorable and require that you use another method such as completing the square or the quadratic formula.

Problem 24.8

A rectangular garden whose width is 4 meters less than its length has an area of 57 square meters. Find the dimensions of the garden to the nearest tenth of a meter.

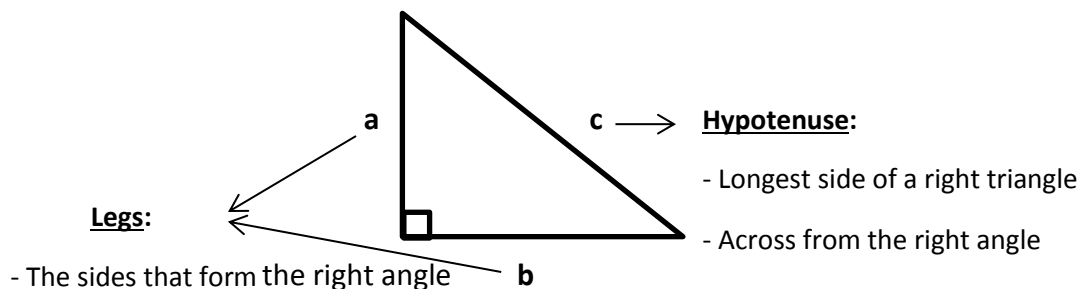
Problem 24.9

A rectangular garden whose length is 5 yards longer than twice its width has an area of 71 square yards. Find the dimensions of the garden to two decimal places.

We may also see problems that involve finding the lengths of the sides in a right triangle. These types of problems will involve the Pythagorean Theorem.

Pythagorean Theorem

$$a^2 + b^2 = c^2$$

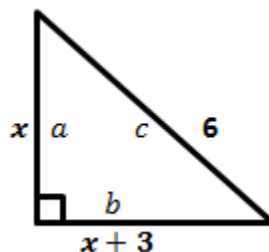


Problem 24.10

The hypotenuse of a right triangle is 6 feet long. One leg is 3 feet longer than the other. Find the length of each leg to the nearest *hundredth* of a foot.

Solution:

It is helpful to draw a diagram and label the sides of the triangle before substituting the information into the Pythagorean Theorem



$$a^2 + b^2 = c^2$$

$$(x)^2 + (x + 3)^2 = 6^2$$

$$x^2 + (x^2 + 6x + 9) = 36$$

$$2x^2 + 6x + 9 = 36$$

$$2x^2 + 6x - 27 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(6) \pm \sqrt{(6)^2 - 4(2)(-27)}}{2(2)}$$

$$x = \frac{-6 \pm \sqrt{36 + 216}}{4}$$

$$x = \frac{-6 \pm \sqrt{252}}{4}$$

$$x = \frac{-6 + \sqrt{252}}{4} \quad \text{or} \quad x = \frac{-6 - \sqrt{252}}{4}$$

$$x \approx 2.47 \quad \text{or} \quad x \approx -5.47$$

$$x + 3 \approx 5.47$$

List the Pythagorean Theorem

Substitute

Simplify and square the binomial $(x + 3)$

Combine like terms

Set quadratic equation equal to zero

List the Quadratic formula

Substitute for a , b and c

$$a = 2; \quad b = 6; \quad c = -27$$

Simplify

Rather than simplifying further, we can use our calculator to obtain a solution rounded to the *nearest hundredth*

We can reject the negative solution because length or width cannot be negative.

Find the length of the other leg

The length of the hypotenuse was already given. One of the legs is about **2.47 feet** long and the other is about **5.47 feet** long (three feet longer).

Problem 24.11

The hypotenuse of a right triangle is 5 feet long. One leg is 4 feet shorter than the other. Find the length of each leg to the nearest *hundredth* of a foot.

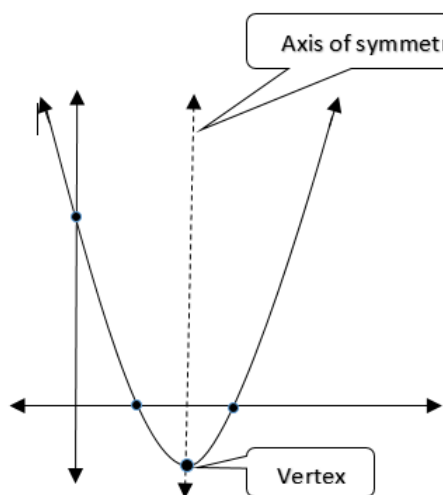
Problem 24.12

The hypotenuse of a right triangle is 1 meter longer than twice the length of one leg. If the other leg is measures 3 meters, find the lengths of each unknown side to the nearest *tenth* of a meter.

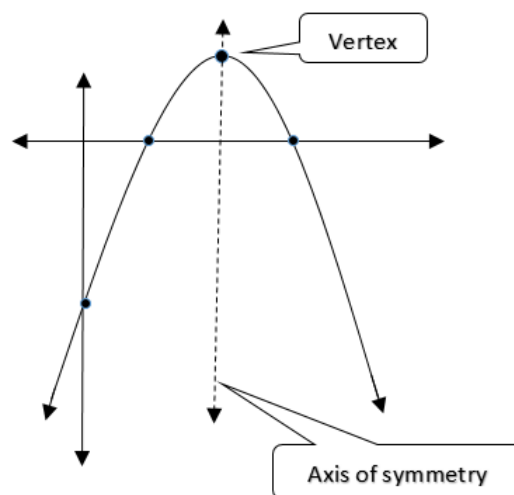
Section 25: Graphing Quadratic Functions

Now that we know how to solve any quadratic equation, we will look at how these methods can aid in the graphing of a quadratic function. You may already be familiar with the shape of a quadratic function.

The graph of a quadratic function
 $f(x) = ax^2 + bx + c$, where $a \neq 0$, is called a **parabola**.



$a > 0$ (parabola opens up)



$a < 0$ (parabola opens down)

We will graph a quadratic function by finding the **coordinates** of the x-intercept(s), the y-intercept and the vertex.

x-intercept(s) – the point or points where the parabola intercepts the x-axis. A quadratic function may have two, one or no x-intercept. We find the x-intercept(s) by solving the quadratic equation $ax^2 + bx + c = 0$ (use previously learned methods of factoring, completing the square, or the quadratic formula)

y-intercept – the point where the parabola intercepts the y-axis. Since quadratics are functions, there can only be *one* y-intercept. We find the y-intercept by evaluation $f(0)$

Axis of Symmetry – the vertical line, splits the parabola in half. The equation of the axis of symmetry is $x = \frac{-b}{2a}$

Vertex – the coordinates of the turning point (the maximum or minimum point). The x-coordinate of the vertex is obtained from the axis of symmetry formula. The y-coordinate is found by evaluating the function at $\frac{-b}{2a}$.

Problem 25.1

Find the coordinates for the vertex of the quadratic function.

$$f(x) = x^2 + 6x + 4$$

Solution

In order to find the coordinates of the vertex, we must first find the axis of symmetry using the formula $x = \frac{-b}{2a}$. This will give us the x-coordinate. We will then use that x-value to find the corresponding y-coordinate.

$$x = \frac{-b}{2a}$$

Formula for the equation of the axis of symmetry

$$x = \frac{-(6)}{2(1)}$$

Substitute 6 for b (the coefficient of x) and 1 for a (the coefficient of x^2)

$$x = \frac{-6}{2} = -3$$

Simplify to obtain a $x = -3$ (the x-coordinate of the vertex)

$$f(-3) = (-3)^2 + 6(-3) + 4$$

Evaluate $f(-3)$ to obtain the y-coordinate

$$f(-3) = 9 - 18 + 4$$

Simplify

$$f(-3) = -5$$

When $x = -3$, $y = -5$

The vertex is the point $(-3, -5)$

Problem 25.2

Find the coordinates for the vertex of the quadratic function.

$$f(x) = -x^2 + 4x - 5$$

Problem 25.3

Find the coordinates for the vertex of the quadratic function.

$$f(x) = 2x^2 - 4x - 3$$

Graphing a quadratic in the form $f(x) = ax^2 + bx + c$

Determine if the parabola opens upwards ($a > 0$) or opens downwards ($a < 0$)

Find the COORDINATES of the x-intercepts by solving $f(x) = 0$. The solutions to the equation $ax^2 + bx + c = 0$ give the x-coordinates for the x-intercepts. The y-coordinate of an x-intercept will always be 0

Find the COORDINATES of the y-intercept by evaluating $f(0)$. This value will give the y-coordinates for the y-intercepts. The x-coordinate of a y-intercept will always be 0

Determine the EQUATION of the axis of symmetry $x = -\frac{b}{2a}$

Determine the COORDINATES of the vertex $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$

Plot the intercepts and vertex. Draw a dashed vertical line for the axis of symmetry.

Problem 25.4

For the function $f(x) = x^2 - 4x - 5$

Determine the **coordinates** of the x-intercepts.

Determine the **coordinates** of the y-intercept.

Determine the **equation** of the axis of symmetry.

Determine the **coordinates** of the vertex.

Sketch a graph of the function, label the x and y axes with an appropriate scale

Solution

This parabola will open upwards since $a > 0$ and will have a minimum value.

To find the x-intercept solve the quadratic equation $x^2 - 4x - 5 = 0$. This quadratic can be factored, but we can also complete the square or use the quadratic formula if preferred.

$$x^2 - 4x - 5 = 0$$

Original quadratic equation

$$(x - 5)(x + 1) = 0$$

Factor the trinomial

$$x - 5 = 0 \quad \text{or} \quad x + 1 = 0$$

Set each factor equal to 0

$$x = 5 \quad \text{or} \quad x = -1$$

Solve

The x-intercepts are (5, 0) and (-1, 0)

State the **coordinates** of the x-intercepts.

NOTE: The y-coordinates will ALWAYS be 0

To find the y-intercept, evaluate $f(0)$.

$$f(x) = x^2 - 4x - 5$$

Original function

$$f(0) = (0)^2 - 4(0) - 5$$

Substitute 0 for x

$$f(0) = -5$$

Simplify

The y-intercept is $(0, -5)$

State the **coordinates** of the y-intercept

NOTE: The x-coordinate will ALWAYS be 0

We will find the **equation** of the axis of symmetry as we did in the previous problems.

$$x = \frac{-b}{2a}$$

Formula for the equation of the axis of symmetry

$$x = \frac{-(-4)}{2(1)} = \frac{4}{2}$$

Substitute for b and a , then simplify

The axis of symmetry is $x = 2$

State the **equation** of the axis of symmetry

Use the x value from the axis of symmetry to find the corresponding y value of the vertex by evaluating $f(2)$.

$$f(x) = x^2 - 4x - 5$$

Original function

$$f(2) = (2)^2 - 4(2) - 5$$

Substitute 2 for x

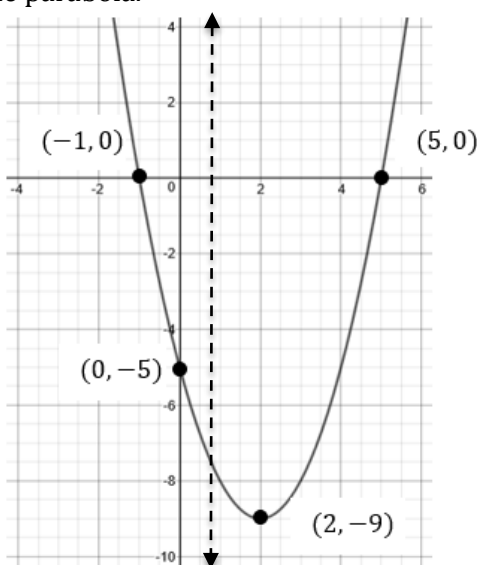
$$f(2) = -9$$

Simplify

The vertex is $(2, -9)$

State the **coordinates** of the vertex

We will now plot the x and y-intercepts, the vertex and the axis of symmetry. Connect the points with a smooth *curve* to form the parabola.



Problem 25.5

For the function $f(x) = x^2 + 6x + 5$

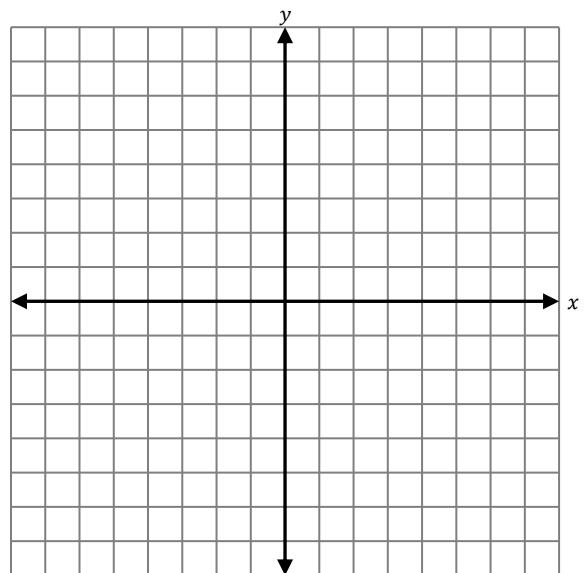
Determine the **coordinates** of the x-intercepts.

Determine the **coordinates** of the y-intercept.

Determine the **equation** of the axis of symmetry.

Determine the **coordinates** of the vertex.

Sketch a graph of the function, label the x and y axes with an appropriate scale



Problem 25.6

For the function $f(x) = x^2 - 6x + 8$

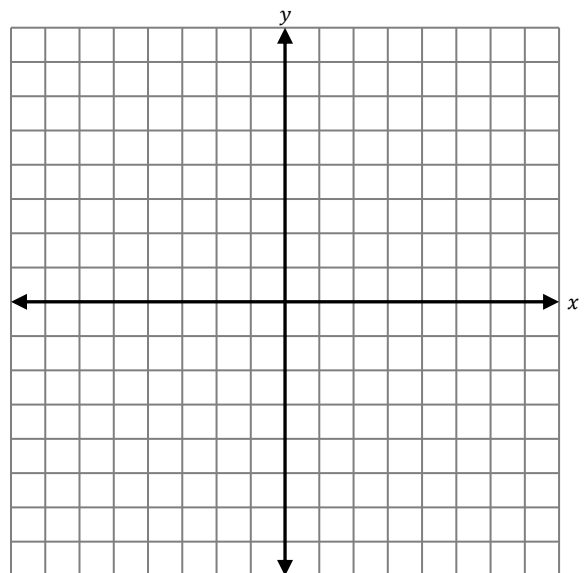
Determine the **coordinates** of the x-intercepts.

Determine the **coordinates** of the y-intercept.

Determine the **equation** of the axis of symmetry.

Determine the **coordinates** of the vertex.

Sketch a graph of the function, label the x and y axes with an appropriate scale



Problem 25.7

For the function $f(x) = 2x^2 - 4x - 6$

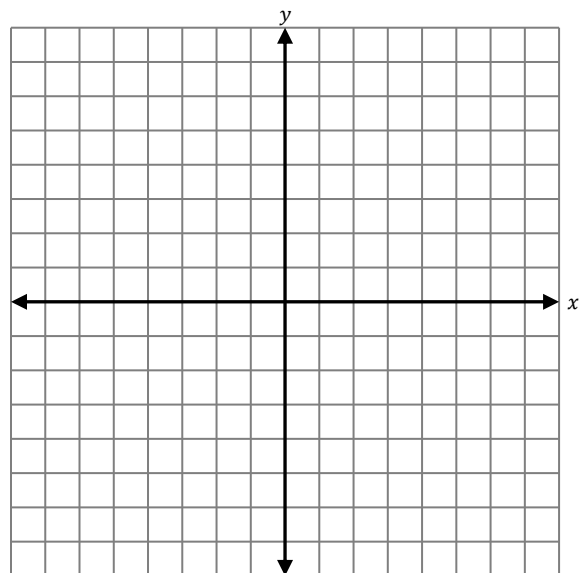
Determine the **coordinates** of the x-intercepts.

Determine the **coordinates** of the y-intercept.

Determine the **equation** of the axis of symmetry.

Determine the **coordinates** of the vertex.

Sketch a graph of the function, label the x and y axes with an appropriate scale



Now let's take a look at an example of a parabola that will open down and has only one x-intercept.

Problem 25.8

For the function $f(x) = -x^2 + 2x - 1$

Determine the **coordinates** of the x-intercepts.

Determine the **coordinates** of the y-intercept.

Determine the **equation** of the axis of symmetry.

Determine the **coordinates** of the vertex.

Sketch a graph of the function, label the x and y axes with an appropriate scale

Solution

This parabola will have a maximum value and open down because $a < 0$

To find the x-intercept solve the quadratic equation $-x^2 + 2x - 1 = 0$. This quadratic can be factored, but we can also complete the square or use the quadratic formula if preferred.

$$-x^2 + 2x - 1 = 0$$

Original quadratic equation

$$-1(x^2 - 2x + 1) = 0$$

Factor out a negative

$$-1(x - 1)(x - 1) = 0$$

Factor the trinomial

$$x - 1 = 0 \quad \text{or} \quad x - 1 = 0$$

Set each factor equal to 0 (the negative is no longer involved in the factoring process)
Solve

$$x = 1 \quad \text{or} \quad x = 1$$

The x-intercept is (1, 0)

State the **coordinates** of the x-intercepts.

To find the y-intercept, evaluate $f(0)$. **Be sure to use the original function.**

$$f(x) = -x^2 + 2x - 1$$

Original function

$$f(0) = -(0)^2 + 2(0) - 1$$

Substitute 0 for x

$$f(x) = -1$$

Simplify

The y-intercept is (0, -1)

State the **coordinates** of the y-intercept

We will find the **equation** of the axis of symmetry as we did in the previous problems.

$$x = \frac{-b}{2a}$$

Formula for the equation of the axis of symmetry

$$x = \frac{-(2)}{2(-1)} = \frac{-2}{-2}$$

Substitute for b and a , **from the original function**, then simplify

The axis of symmetry is $x = 1$

State the **equation** of the axis of symmetry

Use the x value from the axis of symmetry to find the corresponding y value of the vertex by evaluating $f(1)$.

$$f(x) = -x^2 + 2x - 1$$

Original function

$$f(1) = -(1)^2 + 2(1) - 1$$

Substitute 1 for x

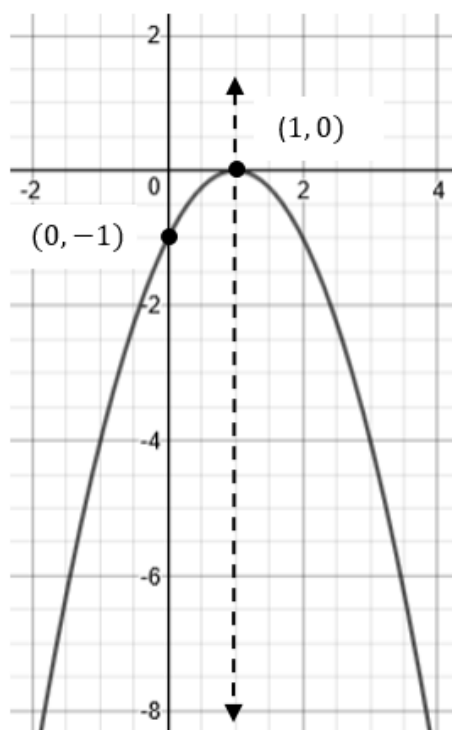
$$f(1) = 0$$

Simplify

The vertex is $(1, 0)$

State the **coordinates** of the vertex

We will now plot the x and y -intercepts, the vertex and the axis of symmetry. Connect the points with a smooth *curve* to form the parabola.



Since there is only one x -intercept, the parabola touches the x -axis at $(1, 0)$.

NOTE: A quadratic with complex solutions will not intercept the x -axis.

Problem 25.9

For the function $f(x) = -x^2 - 6x - 8$

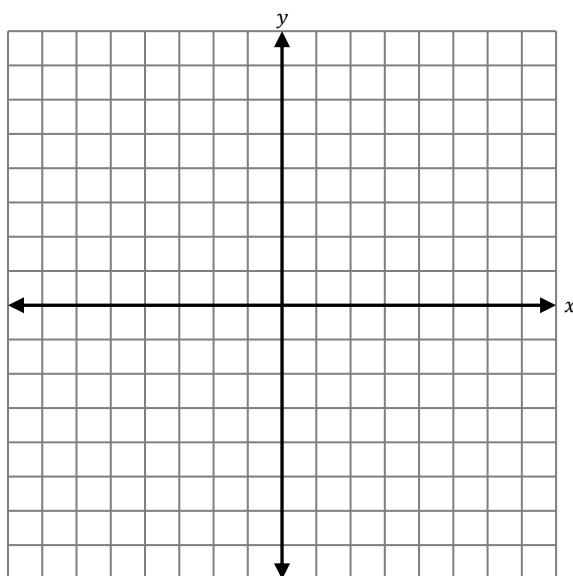
Determine the **coordinates** of the x-intercepts.

Determine the **coordinates** of the y-intercept.

Determine the **equation** of the axis of symmetry.

Determine the **coordinates** of the vertex.

Sketch a graph of the function, label the x and y axes with an appropriate scale



Problem 25.10

For the function $f(x) = x^2 - 6x + 9$

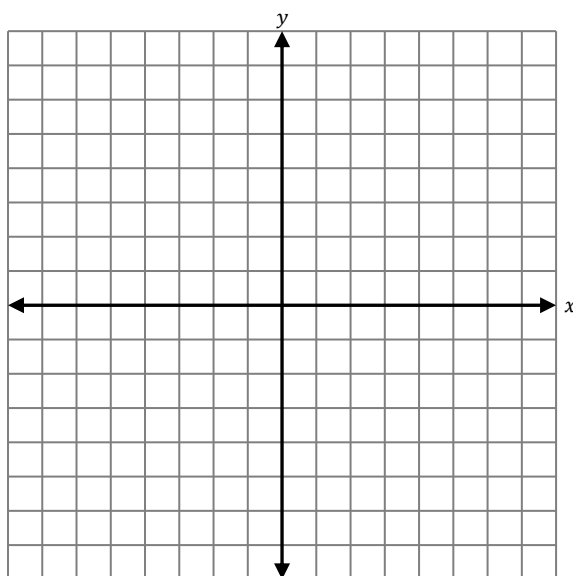
Determine the **coordinates** of the x-intercepts.

Determine the **coordinates** of the y-intercept.

Determine the **equation** of the axis of symmetry.

Determine the **coordinates** of the vertex.

Sketch a graph of the function, label the x and y axes with an appropriate scale



Problem 25.11

For the function $f(x) = x^2 + 3x - 4$

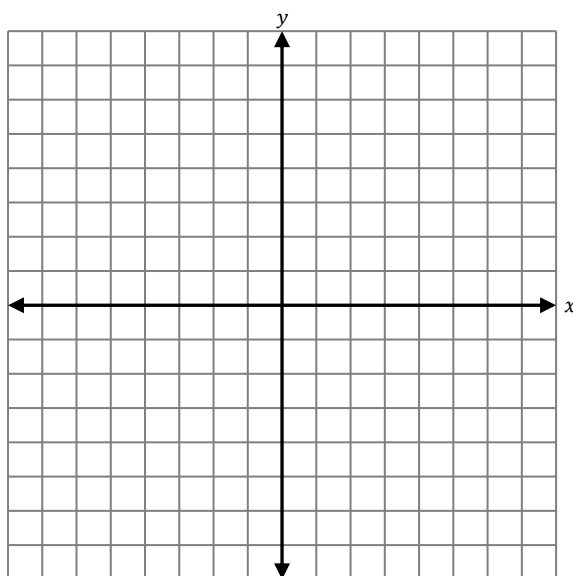
Determine the **coordinates** of the x-intercepts.

Determine the **coordinates** of the y-intercept.

Determine the **equation** of the axis of symmetry.

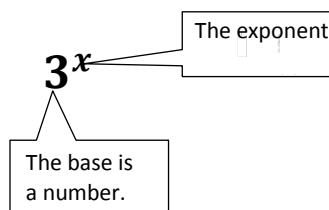
Determine the **coordinates** of the vertex.

Sketch a graph of the function, label the x and y axes with an appropriate scale



Section 26: Exponential and Logarithmic Functions

Up until now, when we have discussed anything having to do with exponents we were considering something like x^2 or y^5 or $a^{\frac{1}{2}}$. Now let's consider what we would call an **exponential function**. It is different in that the *base is a number* and the *exponent is a variable*, such as:



Exponential functions have many practical applications and are used when calculating population growth, radioactive decay, compound interest, the growth of health epidemics, etc.

Definition of an Exponential Function

An exponential function is defined by:

$$f(x) = b^x \text{ or } y = b^x$$

where the base, b , is a constant greater than zero but not 1 ($b > 0, b \neq 1$) and x is any real number.

The following are some examples of exponential functions:

$$f(x) = 2^x \quad g(x) = \left(\frac{1}{2}\right)^x \quad h(x) = 4^{x+2}$$

Notice that in each case the base is a constant number greater than zero and not equal to 1, and that the exponent contains a variable.

Problem 26.1

Identify which of the following are exponential functions.

$$f(x) = 1^x$$

$$g(x) = \left(\frac{1}{3}\right)^{x-1}$$

$$h(x) = 10^{x+3}$$

$$z(x) = (-2)^x$$

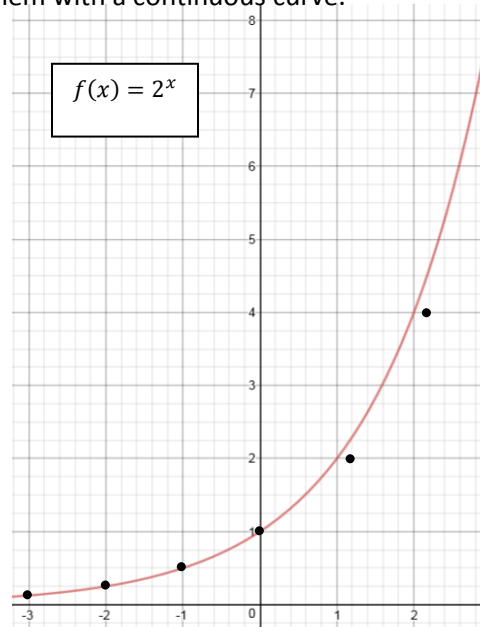
Graphs of Exponential Functions

When $b > 1$:

We begin our exploration of exponential functions with a basic graph. We will use $f(x) = 2^x$. Start by creating a table of values to plot several points and connect them with a continuous curve.

x	$f(x) = 2^x$	(x, y)
-3	$f(-3) = 2^{-3} = \frac{1}{8}$	$(-3, \frac{1}{8})$
-2	$f(-2) = 2^{-2} = \frac{1}{4}$	$(-2, \frac{1}{4})$
-1	$f(-1) = 2^{-1} = \frac{1}{2}$	$(-1, \frac{1}{2})$
0	$f(0) = 2^0 = 1$	$(0, 1)$
1	$f(1) = 2^1 = 2$	$(1, 2)$
2	$f(2) = 2^2 = 4$	$(2, 4)$
3	$f(3) = 2^3 = 8$	$(3, 8)$

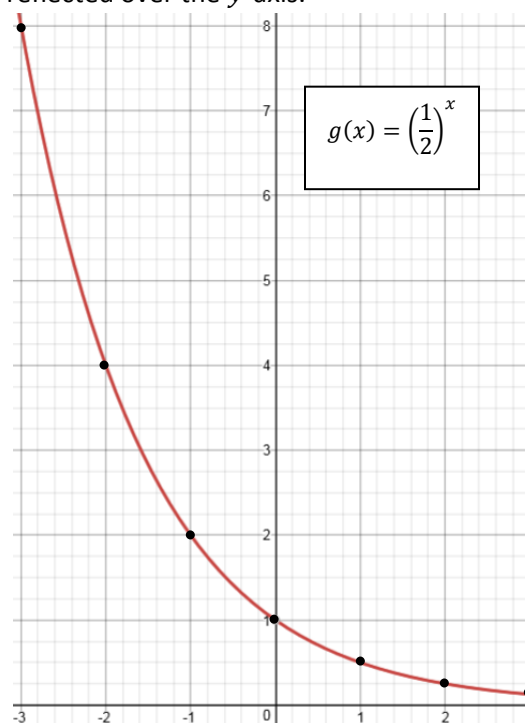
Notice that we are able to substitute any value for x , so the domain of this, and any, exponential function is $(-\infty, \infty)$. Also, the graph lies above the x -axis, so the range is $(0, \infty)$.



When $0 < b < 1$:

Now let's consider $g(x) = (\frac{1}{2})^x$. If we plot the points from a table of values, we can see that the graph looks very similar to the graph of $f(x) = 2^x$, except that it is reflected over the y -axis.

x	$g(x) = (\frac{1}{2})^x$	(x, y)
-3	$g(-3) = (\frac{1}{2})^{-3} = 8$	$(-3, 8)$
-2	$g(-2) = (\frac{1}{2})^{-2} = 4$	$(-2, 4)$
-1	$g(-1) = (\frac{1}{2})^{-1} = 2$	$(-1, 2)$
0	$gf(0) = (\frac{1}{2})^0 = 1$	$(0, 1)$
1	$g(1) = (\frac{1}{2})^1 = \frac{1}{2}$	$(1, \frac{1}{2})$
2	$g(2) = (\frac{1}{2})^2 = \frac{1}{4}$	$(2, \frac{1}{4})$
3	$g(3) = (\frac{1}{2})^3 = \frac{1}{8}$	$(3, \frac{1}{8})$



The domain of this function is also $(-\infty, \infty)$. The graph also lies above the x -axis, so the range is $(0, \infty)$.

Certain situations can be modeled by an exponential function. When an exponential model is given, we will most commonly evaluate for a given time.

Problem 26.2

A car is depreciating according to the formula $V = 32,000(1.96)^{-0.04x}$, where x is the age of the car in years. What is the value of the car when it is five years old?

Solution

We are given the exponential model $V = 32,000(1.96)^{-0.04x}$ and told that the variable x represents the age of the car. V in this situation represents the value of the car for any time, x .

$$V = 32,000(1.96)^{-0.04x}$$

Original exponential model

$$V = 32,000(1.96)^{-0.04(5)}$$

Substitute 5 for x

$$V = 27,970.41$$

Evaluate using a calculator. Be sure to use proper parentheses.

After 5 years, the car will be worth approximately **\$27,970.41**.

Problem 26.3

The amount of money spent in a shopping mall increases according to the formula $A = 42.2(1.39)^{2x}$, where x is the number of hours spent in a mall, ($x \geq 1$). Find the average amount spent after two and a half hours in a mall.

Problem 26.4

A radioactive element decays according to the formula $A = 13.2(2.03)^{-0.003t}$, where t is the number of years and the amount is measured in grams. Find the amount of the element left after 56 years to the nearest hundredth of a gram.

One of the practical applications of exponential functions has to do with compound interest, where banks pay interest on an investment more than once a year (which would be simple interest). There are formulas for two different kinds of compound interest. One type has interest that is compounded n number of times a year. In this case n could represent different compounding periods such as:

Compounding Period	n
Semi-Annually (<i>twice a year</i>)	$n = 2$
Quarterly (<i>4 quarters in a year</i>)	$n = 4$
Monthly (<i>12 months in a year</i>)	$n = 12$
Weekly (<i>52 weeks in a year</i>)	$n = 52$
Daily (<i>365 days a year</i>)	$n = 365$

The formula for n compounding periods per year is:

$$A = P \left(1 + \frac{r}{n} \right)^{nt}$$

where A is the end amount, P is the principal (the start amount), r is the annual interest rate (expressed as a decimal) and t is time (in years).

Problem 26.5

You have \$10,000 to invest in an account that will pay you 6.5% interest per year, compounded quarterly. How much will your investment be worth after 15 years?

Solution

We will be using the formula for n compounding periods per year. First identify what will be substituted into the formula for each variable.

A : This is our unknown (*the amount we end up with*)

P : \$10,000 (*the amount that we start off with*)

r : 6.5% (*interest rate*). Must convert to a decimal: $6.5\% = \frac{6.5}{100} = .065$

n : quarterly, so $n = 4$

t : 15 (number of years)

Now we substitute the values into the formula.

$$A = P \left(1 + \frac{r}{n} \right)^{nt}$$

$$A = 10,000 \left(1 + \frac{.065}{4} \right)^{(4)(15)}$$

$$A = 26,304.71$$

After 15 years, the investment is worth \$26,304.71.

A different formula has interest that is compounded *continuously*. Continuously means that the number of compoundings per year increases without bound. In this formula, we are introduced to a new number, e .

e is an irrational number and should not be confused with a variable. There are many ways of calculating the number e but, since it is irrational, none will give an exact value. If we use the compounding formula from the previous problem and replace P, r and t with 1, we are left with the expression $\left(1 + \frac{1}{n}\right)^n$. To obtain an approximation for e , we observe what happens as n gets larger and larger.

If the table were to continue, the result will approach an irrational number that begins with 2.71828... This number is represented by the symbol e .

e is often called *the natural base* and an exponential function with the natural base e , would be expressed as $f(x) = e^x$.

n	$\left(1 + \frac{1}{n}\right)^n$
1	2.00000
10	2.59374...
50	2.69158...
1,000	2.71692...
10,000	2.71814...
100,000	2.71826...
1,000,000	2.71828...
10,000,000	2.71828...

The formula for continuous compounding is:

$$A = Pe^{rt}$$

where A is the end amount, P is the principal (the start amount), r is the annual interest rate (expressed as a decimal) and t is time (in years).

Problem 26.6

You have \$7,000 to invest in an account that will pay you 7.25% interest per year, compounded continuously. How much will your investment be worth after 10 years?

Solution

We will be using the formula for continuous compounding.

First identify what will be substituted into the formula for each variable.

A : This is our unknown (*the amount we end up with*)

P : \$7,000 (*the amount that we start off with*)

r : 7.25% (*interest rate*). Must convert to a decimal: $7.25\% = \frac{7.25}{100} = .0725$

t : 10 (*number of years*)

$$A = Pe^{rt}$$

$$A = 7000e^{(.0725)(10)}$$

$$A = 14,453.12$$

After 10 years this investment is worth \$14,453.12.

College Algebra through Problem Solving by Cifone, Puri, Maslanko, & Dabkowska.



Problem 26.7

You have \$5,000 to invest in an account that will pay you 8.75% interest per year, compounded monthly. How much will your investment be worth after 8 years?

Problem 26.8

You have \$15,000 to invest in an account that will pay you 5.25% interest per year, compounded continuously. How much will your investment be worth after 7 years?

Problem 26.9

You have \$25,000 to invest in an account that will pay you 9% interest per year, compounded semi-annually. How much will your investment be worth after 5 years?

Problem 26.10

You have \$8,000 to invest in an account that will pay you 4.2% interest per year, compounded continuously. How much will your investment be worth after 10 years?

Logarithmic Functions

Logarithmic functions are *inverse* functions of exponential functions. They essentially “cancel each other out” in the same way as squaring a value and then taking the square root. For this reason, an exponential expression can be expressed in logarithmic form and a logarithmic expression can be expressed in exponential form.

Definition of a Logarithmic Function

For $b > 0, b \neq 1$ and $x > 0$,

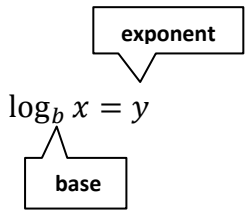
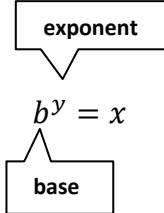
$$f(x) = \log_b x$$

We can read $\log_b x$ as **log base b of x** .

We can convert to and from exponential and logarithmic form using the equivalence


$$\log_b x = y \leftrightarrow b^y = x$$

Let's now examine the parts of these two forms more carefully.

Logarithmic Form	Exponential Form
	


The base in each form may be the easiest to identify. However, notice that when looking at the exponential form, the exponent is y . Since both forms are equivalent, this means that y in logarithmic form *is set equal to an exponent*.

Since we are much more familiar with using exponential form, it is often helpful to change from logarithmic form to exponential form. We know that the logarithmic expression is set equal to y , the exponent. It might help to think about it this way:

$$\log_b x = y \rightarrow b^y = x$$


The **base of the log expression** becomes the **base of the exponential expression** and the value y becomes the exponent of the exponential function. The logarithm has now been “undone” and the exponential expression is set equal to x .

For example:

$$\log_7 x = 2 \rightarrow 7^2 = x$$


In this case, after changing the equation to exponential form, we actually are able to solve and determine that $x = 49$.

We have already discussed the natural base, e . The **natural logarithmic function** could be written as $f(x) = \log_e x$. However, we usually express this function with special notation,

$$f(x) = \ln x$$

This is read “el en of x ”

It is one of two logarithms that are listed on your calculator and looks like:

LN

The other logarithm found on your calculator is the **common logarithmic function** which has a base of 10. This function can be written as $f(x) = \log_{10} x$. Since the number system that we use is the base 10 system, this is an important logarithm and we simply express this function as

$$f(x) = \log x.$$

Notice that it is not necessary to write in the base of 10. The common log is on your calculator as:

LOG

Problem 26.11

Write each logarithmic equation in exponential form. If possible, solve for the variable.

$\log_5 x = 3$	$\log_2 13 = y$
$\log_b 10 = 2$	$\log_{25} 5 = y$
$\log_6 36 = y$	$\ln x = 2$
$\log 100 = x$	$\log x = 3$

Problem 26.12

Write each exponential equation in logarithmic form. (Recall: the logarithmic expression is set equal to the exponent.)

$b^4 = 16$	$e^y = 11$
$13^2 = x$	$81^{\frac{1}{2}} = x$
$10^2 = x$	$10^y = 1$

Inverse Properties of Logarithms

We had already mentioned that logarithmic functions are the inverses of exponential functions. The following are the inverse properties for logarithms and show how these inverses “undo” or cancel each other.

Inverse Properties of Logarithms

For $b > 0, b \neq 1$ and $x > 0$

$$\log_b b^x = x$$

(the logarithm base b of b , raised to some power, equals that power)

$$b^{\log_b x} = x$$

(base b raised to the logarithm with base b , of some number, equals that number)

Problem 26.10

Evaluate.

$$\log_7 7^3$$

Solution

Since the logarithm is in base 7 and the exponential is base 7, we could conclude that since

$$\log_b b^x = x,$$

then $\log_7 7^3 = 3$.

Problem 26.11

Evaluate.

$$13^{\log_{13} 5}$$

Solution

Since the exponential is 13 base 7 and the logarithm is in base 13, we could conclude that since

$$b^{\log_b x} = x,$$

then $13^{\log_{13} 5} = 5$

Problem 26.13

Evaluate.

$$\log_9 9^6$$

Problem 26.14

Evaluate.

$$6^{\log_6 8}$$

Problem 26.15

Evaluate.

$$\ln e^7$$

Problem 26.16

Evaluate.

$$10^{\log 3}$$

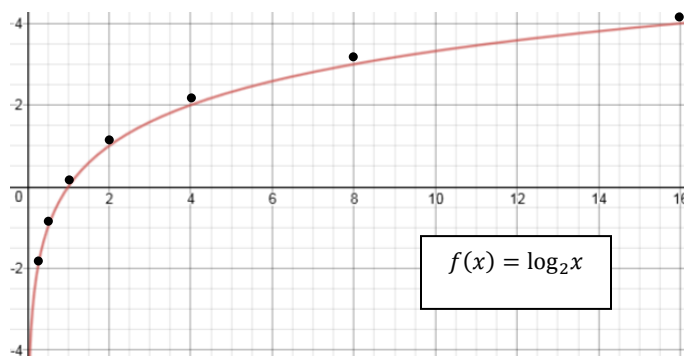
Graphs of Logarithmic Functions

Since logarithms are inverses of exponential functions, their graphs are also inverses of each other. This means that they are reflected over the line $y = x$ (a topic further discussed in pre-calculus!).

When $b > 1$:

Similar to the graphs of exponential functions, we will create a table of values for the function $f(x) = \log_2 x$. Since the base is 2, we will conveniently choose values for x that are powers of 2.

x	$f(x) = \log_2 x$	(x, y)
-1	$f(-1) = \log_2(-1) = \text{error}$	Undefined
0	$f(0) = \log_2(0) = \text{error}$	Undefined
$\frac{1}{4}$	$f\left(\frac{1}{4}\right) = \log_2\left(\frac{1}{4}\right) = -2$	$\left(\frac{1}{4}, -2\right)$
$\frac{1}{2}$	$f\left(\frac{1}{2}\right) = \log_2\left(\frac{1}{2}\right) = -1$	$\left(\frac{1}{2}, -1\right)$
1	$f(1) = \log_2(1) = 0$	(1,0)
2	$f(2) = \log_2(2) = 1$	(2,1)
4	$f(4) = \log_2(4) = 2$	(4,2)
8	$f(8) = \log_2(8) = 3$	(8,3)

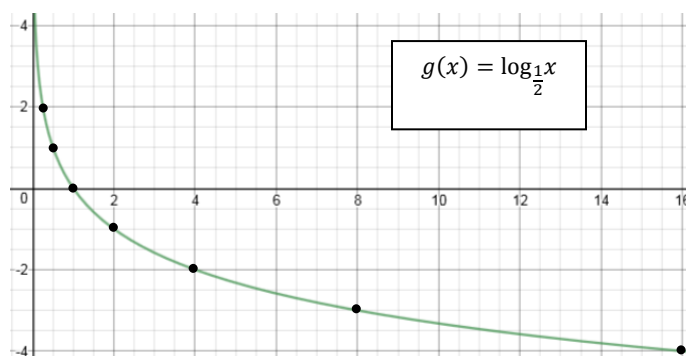


The domain is $(0, \infty)$ and the range is $(-\infty, \infty)$. This is exactly the *inverse* of the domain and range for an exponential function.

When $0 < b < 1$:

Now consider $g(x) = \log_{\frac{1}{2}} x$. The table and graph look very similar $f(x) = \log_2 x$, except it is reflected over the x -axis.

x	$g(x) = \log_{\frac{1}{2}} x$	(x, y)
-1	$g(-1) = \log_{\frac{1}{2}}(-1) = \text{error}$	Undefined
0	$g(0) = \log_{\frac{1}{2}}(0) = \text{error}$	Undefined
$\frac{1}{4}$	$g\left(\frac{1}{4}\right) = \log_{\frac{1}{2}}\left(\frac{1}{4}\right) = 2$	$\left(\frac{1}{4}, 2\right)$
$\frac{1}{2}$	$g\left(\frac{1}{2}\right) = \log_{\frac{1}{2}}\left(\frac{1}{2}\right) = 1$	$\left(\frac{1}{2}, 1\right)$
1	$g(1) = \log_{\frac{1}{2}}(1) = 0$	(1,0)
2	$g(2) = \log_{\frac{1}{2}}(2) = -1$	(2,-1)
4	$g(4) = \log_{\frac{1}{2}}(4) = -2$	(4,-2)
8	$g(8) = \log_{\frac{1}{2}}(8) = -3$	(8,-3)



Again the domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.

It is important to note that the domain of a logarithmic function is always greater than zero. This will become something important to consider when solving logarithmic equations, in Section 28.

Section 27: Properties of Logarithms

We are ready to explore the logarithmic functions in greater detail. We will explore some standard properties of the log function.

The Product Rule

If b , x , and y are positive real numbers, where $b \neq 1$, then

$$\log_b(xy) = \log_b x + \log_b y$$

We can also apply this rule in the other direction when the logarithms have the **same base**.

$$\log_b x + \log_b y = \log_b(xy)$$

Problem 27.1

Expand and simplify, if possible.

(a) $\log_2(4x)$

(b) $\log(1000xy)$

Solutions

(a) Using the product rule, we can split the expression $\log_2(4x)$ into two separate log expression (since there are two factors) both with base 2.

$\log_2(4x)$	Original expression
$= \log_2 4 + \log_2 x$	Apply the product rule
$= 2 + \log_2 x$	Simplify

To simplify $\log_2 4$ you can ask yourself “2 raised to what power will give 4?” So $\log_2 4 = 2$.

(b) Use the Product rule again, but this time there are three factors.

$\log(1000xy)$	Original expression
$= \log 1000 + \log x + \log y$	Apply the product rule
$= 3 + \log x + \log y$	Simplify

To simplify $\log 1000$ we must first recognize that the base is 10, $\log_{10} 1000$. You can ask yourself “10 raised to what power will give 1000?” So $\log 1000 = 3$.

Problem 27.2

Condense and write as a single logarithm. Simplify, if possible.

(a) $\log_3 5 + \log_3 x$

(b) $\log 4 + \log 5 + \log 5$

Solution

(a) Since both expression are of base 3 and are being added, we can use the product rule to condense.

$$\log_3 5 + \log_3 x$$

$$= \log_3(5x)$$

(b) Since both expression are of base 10 and are being added, we can use the product rule to condense and then simplify. We do not need to write the base of 10.

$$\begin{aligned} \log 4 + \log 5 + \log 5 &= \log(4 \cdot 5 \cdot 5) \\ &= \log 100 \end{aligned}$$

$$= 2$$

Problem 27.3

Expand and simplify, if possible.

$$\ln(pq)$$

Problem 27.4

Expand and simplify, if possible.

$$\log_7(49ab)$$

Problem 27.5

Condense and write as a single logarithm. Simplify, if possible.

$$\log_5 8 + \log_5 p$$

Problem 27.6

Condense and write as a single logarithm. Simplify, if possible.

$$\log_6 3 + \log_6 2$$

The Quotient Rule

If b , x , and y are positive real numbers, where $b \neq 1$, and $y \neq 0$, then

$$\log_b \left(\frac{x}{y} \right) = \log_b x - \log_b y$$

We could apply this rule in the other direction when the logarithms have the same base.

$$\log_b x - \log_b y = \log_b \left(\frac{x}{y} \right)$$

Problem 27.7

Expand and simplify, if possible.

(a) $\log_2 \left(\frac{8}{x} \right)$

(b) $\log \left(\frac{1}{y} \right)$

Solution

(a) Using the quotient rule, we can split the expression $\log_2 \left(\frac{8}{x} \right)$ into 2 separate expression, both with base 2.

$\log_2 \left(\frac{8}{x} \right)$	Original expression
$= \log_2 8 - \log_2 x$	Apply the quotient rule
$= 3 - \log_2 x$	Simplify

(b) For this expression the base is 10. However, we do not need to write the base of 10.

$\log \left(\frac{1}{y} \right)$	Original expression
$= \log 1 - \log y$	Apply the quotient rule
$= 0 - \log y$	Simplify
$= -\log y$	

Problem 27.8

Condense and write as a single logarithm. Simplify, if possible.

(a) $\log_3 5 - \log_3 x$

(b) $\log 30 - \log 3$

Solution

(a) Since both expressions are of base 3 and are being subtracted, we can use the quotient rule to condense.

$$\begin{aligned}\log_3 5 - \log_3 x \\ &= \log_3 \left(\frac{5}{x} \right)\end{aligned}$$

(b) In this expression, two natural logarithms are being subtracted and the quotient rule can be applied.

$$\begin{aligned}\ln 30 - \ln 3 &= \ln \left(\frac{30}{3} \right) \\ &= \ln 10\end{aligned}$$

Problem 27.9

Expand and simplify, if possible.

$$\ln \left(\frac{a}{b} \right)$$

Problem 27.10

Expand and simplify, if possible.

$$\log \left(\frac{100}{t} \right)$$

Problem 27.11

Condense and write as a single logarithm. Simplify, if possible.

$$\log_4 13 - \log_4 c$$

Problem 27.12

Condense and write as a single logarithm. Simplify, if possible.

$$\log_3 36 - \log_3 4$$

The Power Rule

If b , x , and r are real numbers, where $b \neq 1$, and $b, x > 0$, then

$$\log_b x^r = r \log_b x$$

We could apply this rule in the other direction.

$$r \log_b x = \log_b x^r$$

Problem 27.13

Expand and simplify, if possible.

(a) $\log_8(x^2y)$

(b) $\log\left(\frac{\sqrt{x}}{y}\right)$

Solution

(a) Since this expression involves the log of a product, we will use the product rule. The power rule will also be used to move the exponent of x to the coefficient of the log.

$\log_8(x^2y)$	Original expression
$= \log_8 x^2 + \log_8 y$	Apply the product rule to separate into two expressions
$= 2 \log_8 x + \log_8 y$	Apply the power rule

(b) Since this expression involves the log of a quotient, we will use the quotient rule. The power rule will also be used once the radical expression is replaced with a rational exponent.

$\log\left(\frac{\sqrt{x}}{y}\right)$	Original expression
$= \log \sqrt{x} - \log y$	Apply the quotient rule
$= \log x^{\frac{1}{2}} - \log y$	Use rational exponents to express \sqrt{x} as $x^{\frac{1}{2}}$
$= \frac{1}{2} \log x - \log y$	Apply power rule to simplify

Problem 27.14

Condense and write as a single logarithm. Simplify, if possible.

(a) $\ln z - 4 \ln x$

(b) $\log_5 x + \log_5(x - 3)$

Solution

(a) Before condensing, all log expressions must have a coefficient of 1 so we will first apply the power rule. Since the log expressions are being subtracted, we will then apply the quotient rule.

$\ln z - 4 \ln x$	Original expression
$= \ln z - \ln x^4$	Apply the power rule to express each log with a coefficient of 1
$= \ln \left(\frac{z}{x^4} \right)$	Apply the quotient rule to condense

(b) Since this expression involves the sum of two log expressions (both with a base of 5), we can use the product rule to condense.

$\log_5 x + \log_5(x + 1)$	Original expression
$= \log_5 x(x + 1)$	All coefficients are 1 so we can apply the product rule
$= \log_5(x^2 + x)$	Simplify, if necessary

Problem 27.15

Expand and simplify, if possible.

$$\ln \left(\frac{e}{z^3} \right)$$

Problem 27.16

Expand and simplify, if possible.

$$\log \left(\frac{\sqrt{x}}{y} \right)$$

Problem 27.17

Condense and write as a single logarithm. Simplify, if possible.

$$2 \log_3 x + \log_3(x + 1)$$

Problem 27.18

Condense and write as a single logarithm. Simplify, if possible.

$$\frac{1}{2} \log t - \log y$$

Change of Base Formula & using the calculator to find logarithms

Most calculators have the natural log (LN) and the common log (LOG) built into the system. However, when we need to calculate a log with a different base, we need to use the change of base formula.

Change of Base Formula

If a , b , and c are positive real numbers different from 1, then,

$$\log_b a = \frac{\log_c a}{\log_c b}$$

Therefore, if we want to use our calculators' built-in log functions, we can either use

$$\log_b a = \frac{\ln a}{\ln b}$$

or

$$\log_b a = \frac{\log a}{\log b}$$

Problem 27.19

Calculate to the nearest ***hundredth***.

(a) $\log_2 7$

(b) $\frac{\log_3 13}{4}$

Solution

We will use the common log for all our calculations (but you may use the natural log if you wish).

(a)

$$\log_2 7 = \frac{\log 7}{\log 2}$$

Use the change of base formula.

$$= 2.80735492$$

Use the calculator to find the value

$$= 2.81$$

Round to the nearest hundredth.

NOTE: Be sure to use parentheses when necessary in your calculator!

(b)

$$\frac{\log_3 13}{4} = \frac{\left(\frac{\log 13}{\log 3}\right)}{4}$$

$$= 0.58367937$$

$$= 0.58$$

Use the change of base formula in the denominator.

Use the calculator to find the value.

Round to the nearest hundredth.

Problem 27.20

Calculate to the nearest ***hundredth***.

$$\log_3 56$$

Problem 27.21

Calculate to the nearest ***hundredth***.

$$\log_{1.8} 3.5$$

Problem 27.22

Calculate to the nearest ***tenth***.

$$\frac{\log_2 67}{5}$$

Problem 27.23

Calculate to the nearest ***tenth***.

$$\frac{\log_{2.125} 1.45}{4}$$

Section 28: Exponential & Logarithmic Equations

Now, we will use our knowledge of exponential and logarithmic functions and the properties of logarithms to solve exponential and logarithmic equations.

Exponential Equations

In some exponential equations, it is possible to express both sides of the equation with the same base. If this can be accomplished, then we can simply set the exponents equal to each other and solve for the variable.

Exponential Equations with the SAME BASE

$$\text{If } b^x = b^y, \text{ then } x = y$$

Problem 28.1

Solve for x .

(a) $3^x = 81$

(b) $2^x = \sqrt{2}$

(c) $5^{x+1} = \frac{1}{5}$

Solutions

(a) We can write 81 in terms of the base 3. Then set the exponents equal.

$$3^x = 81$$

$$3^x = 3^4$$

$$x = 4$$

(b) We rewrite $\sqrt{2}$ in terms of the base 2 using rational exponents. Then set the exponents equal.

$$2^x = \sqrt{2}$$

$$2^x = 2^{\frac{1}{2}}$$

$$x = \frac{1}{2}$$

(c) Again, we can rewrite $\frac{1}{5}$ in terms of the base 5 using negative exponents. Then set the exponents equal and solve for x .

$$5^{x+1} = \frac{1}{5}$$

$$5^{x+1} = 5^{-1}$$

$$x + 1 = -1$$

$$x = -2$$

Sometimes, it is not possible to write both sides in terms of the same base. We will use the inverse function of logarithms to help us get to the solution.

Problem 28.2

Solve for x . Round your answer to the nearest hundredth.

(a) $2^x = 5$

(b) $4^{3x+1} = 13$

(c) $5(1.23)^{4x} = 18$

Solutions

(a) We will convert the equation into its equivalent logarithmic form and then solve using our calculator.

$2^x = 5$	Original equation
$x = \log_2 5$	Convert to logarithmic form.
$x = \frac{\log 5}{\log 2}$	Use the change of base formula to plug into the calculator.
$x = 2.32192809$	
$x = 2.32$	Round to the nearest hundredth.

(b) Once converted to log form, this equation involves additional steps to solve for x .

$4^{3x+1} = 13$	Original equation
$3x + 1 = \log_4 13$	Convert to logarithmic form.
$3x = (\log_4 13) - 1$	Subtract 1 from each side. The parentheses help to show that 1 can NOT be subtracted from the 13 in the log expression
$x = \frac{(\log_4 13) - 1}{3}$	Divide by 3.
$x = \frac{\left(\frac{\log 13}{\log 4}\right) - 1}{3}$	Use the change of base formula and plug into the calculator.
$x = 0.28340661$	
$x = 0.28$	Round to the nearest hundredth.

(c) When necessary, we must first **isolate the exponential expression** before converting to logarithmic form.

$$5(1.23)^{4x} = 18$$

$$\frac{5(1.23)^{4x}}{5} = \frac{18}{5}$$

$$(1.23)^{4x} = 3.6$$

$$4x = \log_{1.23} 3.6$$

$$x = \frac{\log_{1.23} 3.6}{4}$$

$$x = \frac{\log 3.6}{4 \log 1.23}$$

$$x = 1.546915664$$

$$x = 1.55$$

Original equation

Divide each side by 5 to **isolate the exponential expression**.

Convert to logarithmic form.

Divide by 4.

Use the change of base formula and plug into the calculator.

Round to the nearest hundredth.

Problem 28.3

Solve for x .

$$49^x = 7$$

Problem 28.4

Solve for x to the nearest hundredth.

$$6^x = 18$$

Problem 28.5

Solve for t to the nearest hundredth.

$$(1.0225)^{4t} = 2.55$$

Problem 28.6

Solve for x to the nearest hundredth.

$$5(4^{2x+3}) = 10$$

Problem 28.7Solve for x to the nearest hundredth.

$$e^x + 2 = 8$$

Problem 28.8Solve for x to the nearest hundredth.

$$21^{3x} = 5$$

Now, we will apply the above techniques to solve a compound interest problem where the unknown is part of the exponent.

Problem 28.9

How long will it take for an investment of \$2,000 to grow to \$4,500 at 5.5% annual interest compounded quarterly? Use $A = P \left(1 + \frac{r}{n}\right)^{nt}$. Round to the nearest tenth of a year.

Solution

First, we will list the known values for each variable.

A : \$4,500 (*the amount we end up with*)

P : \$2,000 (*the amount that we start off with*)

r : 5.5% (*interest rate*). Must convert to a decimal: $5.5\% = \frac{5.5}{100} = .055$

n : quarterly, so $n = 4$

t : this is the unknown (number of years)

$$4,500 = 2,000 \left(1 + \frac{0.055}{4}\right)^{4t}$$

$$\frac{4,500}{2,000} = \frac{2,000}{2,000} \left(1 + \frac{0.055}{4}\right)^{4t}$$

$$2.25 = (1.01375)^{4t}$$

$$4t = \log_{1.01375} 2.25$$

$$t = \frac{\log_{1.01375} 2.25}{4}$$

$$t = 14.84532131$$

$$t = 14.8 \text{ years}$$

Plug each known value into the formula.

Isolate the exponential expression by
Dividing each side by 2,000

Divide and simplify.

Convert to logarithmic form.

Divide each side by 4.

Use the change of base formula and plug into the calculator.

Round to the nearest tenth of a year.

Problem 28.10

How long will it take for an investment of \$3,000 to grow to \$5,700 at 4.5% annual interest compounded semi-annually? Use

$A = P \left(1 + \frac{r}{n} \right)^{nt}$. Round to the nearest tenth of a year.

Problem 28.11

How long will it take for an investment of \$4,000 to grow to \$6,500 at 6.5% annual interest compounded quarterly? Use

$A = P \left(1 + \frac{r}{n} \right)^{nt}$. Round to the nearest tenth of a year.

Logarithmic Equations

In order to solve logarithmic equations, we will convert into exponential form and then solve for the variable. It is very important to check all solutions to a logarithmic equation to be sure that the value inside each logarithmic expression (the argument) is greater than zero.

Problem 28.12

Solve for x and check whether your answer is viable.

(a) $\log_3 x = 5$

(b) $\log_5(2x + 3) = 2$

Solutions

(a) In this basic logarithmic equation, once we convert to exponential form, the solution will be obvious. We should still check to make sure that we are taking the logarithm of a positive number.

$$\log_3 x = 5$$

$$x = 3^5$$

$$x = 243$$

Original equation

Convert to exponential form.

Simplify.

CHECK:

$$\log_3 x = 5$$

$$\log_3(243) \neq 5$$

$$5 = 5 \checkmark$$

(b) Once converted to exponential form, we must solve a linear equation for x . Remember to check the solution.

$$\log_5(2x + 3) = 2$$

Original equation

$$2x + 3 = 5^2$$

Convert to exponential form.

$$2x + 3 = 25$$

Simplify and solve for x .

$$2x = 22$$

Subtract 3 from each side.

$$x = 11$$

Divide each side by 2.

CHECK:

$$\log_5(2x + 3) = 2$$

$$\log_5(2(11) + 3) \neq 2$$

$$\log_5(22 + 3) \neq 2$$

$$\log_5(25) \neq 2$$

$$2 = 2 \checkmark$$

In the following examples, we must first apply the product or quotient rule to condense the logarithmic expressions. Once there is a single logarithmic expression, we can convert to exponential form to solve.

Problem 28.13

Solve for x and check whether your answer is viable.

(a) $\log_6 x + \log_6(x - 5) = 2$

(b) $\log x - \log(x - 9) = 1$

Solutions

(a) Since addition is involved and both expressions have a base of 6, we can apply the product rule.

$$\log_6 x + \log_6(x - 5) = 2$$

Original equation

$$\log_6(x(x - 5)) = 2$$

Use product rule to convert to a single logarithm.

$$x(x - 5) = 6^2$$

Convert to exponential form.

$$x^2 - 5x = 36$$

Simplify by distributing.

$$x^2 - 5x - 36 = 0$$

Set the equation equal to zero

$$(x - 9)(x + 4) = 0$$

Factor.

$$x - 9 = 0, x + 4 = 0$$

Solve for x .

$$x = 9, x = -4$$

Check BOTH solutions.

CHECK:

$\log_6 x + \log_6(x - 5) = 2$ $\log_6(9) + \log_6((9) - 5) = 2$ $\log_6 9 + \log_6 4 = 2$ $\log_6 36 = 2$ $2 = 2$	$\log_6 x + \log_6(x - 5) = 2$ $\log_6(-4) + \log_6((-4) - 5) = 2$ $\log_6(-4) + \log_6(-9) = 2$ $x \neq -4 \text{ Because we cannot take the}$ $\text{log of a negative number}$
--	--

(b) Since subtraction is involved and both expressions have a base of 10, we can apply the quotient rule.

$$\log x - \log(x - 9) = 1$$

Original equation

$$\log_{10} \left(\frac{x}{x - 9} \right) = 1$$

Use quotient rule to convert to a single logarithm.

$$\frac{x}{x - 9} = 10^1$$

Convert to exponential form.

$$x = 10(x - 9)$$

Multiply each side by the LCD $(x - 9)$.

$$x = 10x - 90$$

Distribute.

$$-9x = 90$$

Subtract $10x$ from each side.

$$x = 10$$

Divide each side by -9 .

Problem 28.14

Solve for x and check whether your answer is viable.

$$\log_2 x = 7$$

Problem 28.15

Solve for x and check whether your answer is viable.

$$\log_x 343 = 3$$

Problem 28.16

Solve for x and check whether your answer is viable.

$$\log_4(6x + 4) = 3$$

Problem 28.17

Solve for x and check whether your answer is viable.

$$\log_7(10x - 1) = 2$$

Problem 28.18

Solve for x and check whether your answer is viable.

$$\log_2 x + \log_2 5 = 3$$

Problem 28.19

Solve for x and check whether your answer is viable.

$$\log_3 x + \log_3(x + 6) = 3$$

Problem 28.20

Solve for x and check whether your answer is viable.

$$\log_8 x + \log_8(x + 12) = 2$$

Problem 28.21

Solve for x and check whether your answer is viable.

$$\log_6(3x) - \log_6(x - 2) = 1$$